Magnetic energy, multiply connected domains, and force-free fields

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This paper addresses the question of magnetic energy in mutiply connected domains. It is shown that the magnetic energy must in general include a boundary term that is usually assumed to vanish. The physical interpretation of this term is discussed in terms of de Rham's theorems.

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INTRODUCTION

The discovery of high-temperature superconductors has stimulated renewed interest in force-free magnetic-field configurations [1,2]. Determining the magnetic energy of such configurations involves some subtleties that at first may not be apparent. The question of the magnetic energy contained in force-free fields was initially addressed by Chandrasekhar and Woltjer [3]. It was later proved by Woltjer [4], and under somewhat less restrictive conditions by Moffatt [5], that the equations of nondissipative magnetohydrodynamics have the magnetic helicity

$$I=\int_V \mathbf{A}\cdot\mathbf{B}\,dV \tag{1}$$

as a constant of the motion. The helicity has subsequently been identified with the asymptotic Hopf invariant [6]. Other such integrals have been discussed by Woltjer [7] and Kruskal and Kulsrud [8]. In minimizing the total magnetic energy with the constraint that helicity be conserved, Woltjer introduced a constant α through the method of Lagrangian multipliers, to show that the variational problem

$$\delta\int_V \{|\nabla\times\mathbf{A}|^2-\alpha\,\mathbf{A}\cdot(\nabla\times\mathbf{A})\}dV=0 \tag{2}$$

implies that the force-free magnetic-field equations

$$\nabla\times\mathbf{B}=\alpha\mathbf{B} \tag{3}$$

are satisfied. In this way, Woltjer succeeded in showing that force-free fields with constant α correspond to the lowest state of magnetic-field energy that a closed system may attain [9]. Freedman [10] has addressed the issue of a lower bound to the energy.

Taylor [11], in discussing the relaxation of a toroidal plasma, pointed out that for the case where the field generates magnetic surfaces and where the conductivity is assumed to be perfect, there is one invariant of the form given by Eq. (1) for each surface. He then noted that the magnetic field that results from minimizing the energy, for all variations $\delta\mathbf{A}$ which leave this set of invariants unchanged, satisfies

$$\nabla\times\mathbf{B}=\alpha(x)\mathbf{B}\ , \tag{4}$$

where α is now a function of position, but is constant on each of the magnetic surfaces. The latter assertion becomes apparent by taking the divergence of Eq. (4) to obtain $\mathbf{B}\cdot\nabla\alpha=0$, which shows that α is a constant on a field line. If the field line is constrained to a surface, α is constant on that surface [12].

Magnetic fields do not, in general, form magnetic surfaces. Such surfaces arise in magnetohydrostatic equilibria and for some highly symmetric field configurations. When the field does form magnetic surfaces, Cowling's theorem [13] tells us that they cannot be simply connected. In addition, there is a theorem by Hopf [14] that states that the torus and the Klein bottle are the only smooth, compact, connected surfaces without boundary that can have a nonsingular (nowhere vanishing) vector field. Thus, nonsingular magnetic surfaces can be expected to have the topology of nested tori [8]. Arnold [6] has shown that if $\mathbf{B}$ is a divergence-free field on a three-dimensional closed orientable Riemannian manifold which satisfies $\mathbf{B}\times(\nabla\times\mathbf{B})=\nabla\psi$, then the field lines of $\mathbf{B}$ and $\nabla\times\mathbf{B}$ lie on tori that are given by ψ=const. When $\nabla\psi=0$ (the force-free case), the field lines will also lie on two-dimensional tori, provided the field is nonsingular and α is not constant. If α is constant, the field can have more complicated topologies [15].

This paper examines the question of magnetic-field energy for nonsingular toroidal configurations in the approximation of ideal, static magnetohydrodyamics. The discussion makes use of de Rham's first theorem, an exposition of which can be found in Flanders [16] and Fenn [17]. An extensive discussion of the content of de Rham's theorems has also been given by Blank, Friedrichs, and Grad [18].

MAGNETIC ENERGY

The total magnetic-field energy associated with currents $\mathbf{J}$ in a volume V is given by

$$E=\tfrac{1}{2}\int_V \mathbf{J}\cdot\mathbf{A}\,dV\ . \tag{5}$$

Taking the dot product of Eq. (4), written in terms of $\mathbf{H}$, with $\mathbf{A}$ and integrating results in [19]

$$E=\tfrac{1}{2}\int_V \mathbf{H}\cdot(\nabla\times\mathbf{A})dV+\tfrac{1}{2}\int_S(\mathbf{H}\times\mathbf{A})\cdot d\mathbf{S}$$
$$=\tfrac{1}{2}\int_V \alpha\,\mathbf{A}\cdot\mathbf{H}\,dV\ , \tag{6}$$

where S is the bounding surface of the volume V. It is assumed here, and throughout the remainder of this paper, that $\mathbf{B}=\mu_0\mathbf{H}$. The first equality in Eq. (6) is true in general, while the second is true for force-free fields. For constant-α fields, Yang [20] has noted that the magnetic energy can be written as the product of α and the total magnetic helicity. If α is not assumed to be constant, such a simple decomposition is not possible.

From Eq. (6), it can be seen that both the volume and the surface integrals contribute to the magnetic energy. The usual textbook discussion of the surface integral in Eq. (6) takes the bounding surface of the configuration to be at infinity. Because the source currents are assumed to occupy a finite, closed volume and since $\mathbf{H}$ falls off at least as fast as r^{-2} and $\mathbf{A}$ as r^{-1}, it is argued that the integral vanishes as r^{-1} at a minimum. However, if S is a magnetic surface of finite dimension, this argument does not hold and the integral cannot be ignored. That this integral must play a role in the energy of finite force-free fields has been noted by Reiman [21], who pointed out that if account is not taken of this boundary term, one obtains an incorrect expression for the energy of force-free state. If the currents are confined to a volume V_c and if the whole space V_∞ is simply connected, the physical interpretation of the boundary term is readily understood by observing that it is identically equal to

$$\int_{V_\infty - V_c} \mathbf{B}\cdot\mathbf{H}\, dV$$

transformed by Stokes' theorem to a surface integral; it is the energy in the field exterior to V_c due to currents within V_c.

If the domain is simply connected, the integrals of Eq. (6) are gauge invariant, as would be expected since the energy must be gauge invariant. This can be seen as follows: (1) Consider first the surface integral. Under the transformation $\mathbf{A}\rightarrow \mathbf{A}+\nabla\chi$, the integral associated with $\nabla\chi$ is

$$\int_S (\mathbf{H}\times\nabla\chi)\cdot d\mathbf{S} = \int_V \nabla\cdot(\mathbf{H}\times\nabla\chi)dV = \int_V \nabla\cdot(\chi\mathbf{J})dV = \int_S \chi\mathbf{J}\cdot\hat{\mathbf{n}}\, dS = 0 \ , \quad (7)$$

where $\nabla\cdot\mathbf{J}=0$ has been used. The last equality is true in general if S bounds the currents and is true for force-free fields in particular since $\mathbf{J}=\alpha\mathbf{H}$ lies in the magnetic surface S. (2) The integral on the right-hand side of Eq. (6) is also gauge invariant since under a gauge transformation, the integral associated with $\nabla\chi$ is

$$\int_V \alpha\nabla\chi\cdot\mathbf{H}\, dV = \int_V \nabla\cdot(\alpha\chi\mathbf{H})dV = \int_S \alpha\chi\mathbf{H}\cdot\hat{\mathbf{n}}\, nS = 0 \ . \quad (8)$$

Note that in simply connected domains, the helicity is also invariant under a gauge transformation provided $\mathbf{H}\cdot\hat{\mathbf{n}}=0$.

In a multiply connected domain (for example, the interior of a toroid), the meaning of the terms involving $\mathbf{A}$ in Eq. (6) is not immediately clear since, under a gauge transformation, the function χ is not necessarily single valued. Yet, for the energy to be well defined in such domains, it must be gauge invariant. The role of multivalued gauge transformations will be discussed later in terms of two physical examples: a simple ring current and a force-free configuration, both in a toroidal domain.

To discuss the magnetic-field energy in multiply connected domains, it will be useful to recast Eqs. (5) and (6) into the language of differential forms in three-dimensional Euclidean space. This will facilitate contact being made with de Rham's theorems. To that end, replace the vector potential $\mathbf{A}$ with a 1-form A. Then the magnetic field $\mathbf{B}$ is given by the 2-form $B=dA$ (equivalent to the local statement $\mathbf{B}=\nabla\times\mathbf{A}$). Taking the exterior derivative of B shows that B is a closed 2-form, i.e., $dB=0$ (corresponding to $\nabla\cdot\mathbf{B}=0$). B is also *locally* an exact form since $B=dA$. The question of the existence of A *globally* is the subject of de Rham's theorems.

Expressing Eqs. (5) and (6) in the language of differential forms also requires the introduction of the Hodge $*$ or duality operator. Here, since the application is to three-dimensional space, the duality operator maps p-forms onto $(3-p)$-forms. Note that this operator has the property that the subspace of $*\omega$ orthogonal to that of ω, where ω is a p-form. Although the $*$ operator is defined locally, it is independent of local coordinates, but does depend on the existence of an inner product and the orientation of the space.

The helicity, Eq. (1), is now readily expressed in terms of differential forms as

$$I=\int_V A\wedge B \ . \quad (9)$$

If the flux across every closed surface in the domain V vanishes, this is the Hopf invariant mentioned in the Introduction. If the current $\mathbf{J}$ is defined as the 1-form J, the 2-form $*J$ can be used to write the differential form of Ampere's circuital law, $\mu_0 *J=d(*B)$. The magnetic energy, Eq. (5), can be written as

$$\mu_0 E=\tfrac{1}{2}\int_V A\wedge d(*B) \ . \quad (10)$$

Using Stokes' theorem and the force-free relation $d(*B)=\alpha B$, Eq. (6) becomes

$$\mu_0 E=\tfrac{1}{2}\int_V B\wedge *B-\tfrac{1}{2}\int_{\partial V} A\wedge *B=\tfrac{1}{2}\int_V \alpha A\wedge B \ . \quad (11)$$

Notice that if the inner product on p-forms is defined as

$$\langle\alpha,\beta\rangle=\int_V \alpha\wedge *\beta \ , \quad (12)$$

the boundary term can be written as [22]

$$\int_{\partial V} A\wedge *dA=\langle B,B\rangle-\langle A,\delta dA\rangle \ . \quad (13)$$

This shows that unless the boundary term vanishes, d and δ [the codifferential operator given in three-dimensional space by $\delta\omega=(-1)^p *d(*\omega)$, ω being a p-form] will not be adjoint operators, as is usually the case for closed manifolds. Note that the codifferential operator can be used to write the differential form of Ampere's circuital law as $\mu_0 J=\delta B$ since $**=+1$ applied to 1-forms. The inner product $\langle B,B\rangle$ is positive definite, and in terms of classical notation is $\int_V B^2 dV$.

de RHAM'S THEOREMS

de Rham's theorems are expressed here in the form given by Flanders [16]. If ω is a closed p-form, for each p-cycle [23] $\mathbf{z}$, one can define a "period" of ω by $\int_{\mathbf{z}}\omega$. The period only depends on the homology class of $\mathbf{z}$. If $\mathbf{z}$ is a boundary, then by Stokes' theorem the period vanishes. Therefore, if $\sum_i a_i \mathbf{z}_i =$boundary, then $\sum_i a_i \int_{\mathbf{z}_i}\omega = 0$. de Rham's first and second theorems are then expressible as follows: (i) A closed p-form is exact if and only if all its periods vanish, and (ii) if each p-cycle $\mathbf{z}$ is assigned a number per ($\mathbf{z}$), there is a closed p-form ω which has the assigned periods $\int_{\mathbf{z}}\omega = \text{per}(\mathbf{z})$ for each p-cycle $\mathbf{z}$ [subject to the consistency relation $\sum_i a_i \text{per}(\mathbf{z}_i) = 0$ if $\sum_i a_i \mathbf{z}_i$ is a boundary].

If ω is set equal to the 2-form B, the substance of de Rham's first theorem (the one that will be most used here) is as follows: Since the magnetic field B is solenoidal ($dB = 0$), a vector potential A exists locally such that $B = dA$, i.e., B is locally exact. For A to exist globally [24], the closed surface periods $\int_{\mathbf{z}} B$ must vanish ($\mathbf{z}$ being a 2-cycle). The nonexistence of magnetic monopoles guarantees that this will always be the case.

Even though the closed-surface periods vanish, it is interesting to supplement the solenoidal condition for $\mathbf{B}$ with the boundary condition that the normal component of $\mathbf{B}$ vanish [25]. While this condition alone would imply that the closed-surface periods vanish (even if magnetic monopoles existed), it also allows one to use the concept of relative homology to define *open*-surface periods, in terms of the homology modulo the boundary [26], which can be interpreted as fluxes. If Σ and Σ' are such that $\Sigma' \sim \Sigma \pmod{\partial D}$, where ∂D is the boundary of the domain D, then $\Sigma - \Sigma'$ can be made into a closed bounding surface by the addition of part of the boundary. The integral $\int_\Sigma \mathbf{B}\cdot\hat{\mathbf{N}}\, d\Sigma = \int_\Sigma B$ is then a period. Interestingly enough, the converse is also true, as pointed out by Blank, Friedrichs, and Grad [18]. If the open-surface integrals $\int_\Sigma B$ only depend on the homology class (if they are periods), then the closed-surface periods of $\mathbf{B}$ vanish and the normal component of $\mathbf{B}$ also vanishes on the boundary.

It is also possible to define the homology of open arcs modulo the boundary. Two points are said to be homologous if they bound an arc. If these points are on the boundary of D, then two arcs, C and C' contained in D, are said to be homologous modulo the boundary, $C \sim C' \pmod{\partial D}$, if their 0-dimensional boundaries are homologous in $\bar{D}$. Thus, when $C \sim C' \pmod{\partial D}$, it is possible to complete $C - C'$ to a closed, bounding curve in D by adding arcs on the boundary [27].

In what follows, various integrals involving the scalar or vector potential will be evaluated by introducing cuts on open surfaces Σ or along open arcs.

INTERPRETATION OF THE BOUNDARY TERM

With reference to Fig. 1, let the domain D be the exterior of a toroidal surface S_1, bounded by S_0 and S_1. Assume that there exists a 1-form X such that $dX = 0$ and a

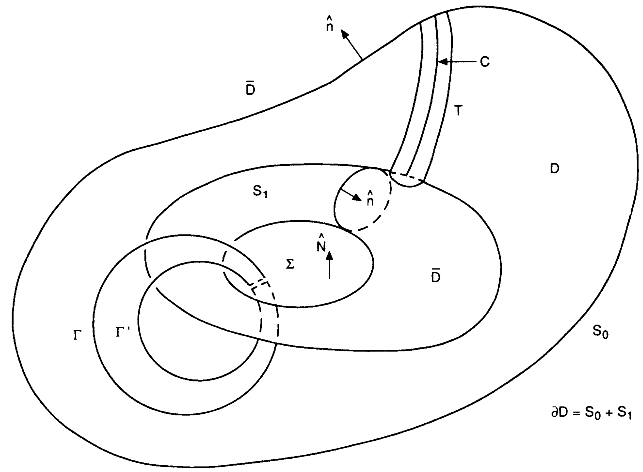

FIG. 1. The domain D exterior to a toroidal surface S_1 bounded by S_0 and S_1. The open surface Σ and the open arc C are taken as cuts to make ϕ and A, respectively, single valued.

2-form Y such that $dY = 0$. Then there exist locally a 0-form (function) ϕ such that $X = d\phi$ and a 1-form A such that $Y = dA$. ϕ can be made single valued by taking the open surface Σ as a cut. Consider the integral $\int_D X \wedge Y$. Using the relationship

$$d(\phi Y) = d\phi \wedge Y + \phi\, dY \,, \tag{14}$$

this integral can be written as

$$\int_D X \wedge Y = \int_D d(\phi Y) = \int_{\partial D} \phi Y + \int_\Sigma [\phi] Y \,, \tag{15}$$

where $[\phi] = \int_\Gamma d\phi = \int_\Gamma X$ is the jump across Σ. This jump is independent of the path Γ chosen provided Γ circles the torus. To see this, note that the two paths Γ and Γ' of Fig. 1 can be connected along both sides of the open-surface cut Σ (as indicated). If one integrates along the closed path consisting of these connections and Γ and Γ', the contributions of the connections on either side of Σ will cancel. Stokes' theorem then allows the integral over this closed path to be replaced by the integral of dX over the enclosed surface bounded by $\Gamma - \Gamma'$, which vanishes since $dX = 0$. Thus, taking account of the direction of integration along Γ and Γ', the integral over Γ is equal to that over Γ' and $[\phi]$ is independent of the choice of Γ showing that $[\phi]$ is constant on Σ. This means $[\phi]$ can be removed from under the integral and Eq. (15) can be written as

$$\int_D X \wedge Y = \int_{\partial D} \phi Y + \int_\Gamma X \int_\Sigma Y \,. \tag{16}$$

Alternatively, one can introduce the open are C (connecting the component of $\bar{D}$ inside the torus with that outside S_0) as a cut so as to make A single valued. Then, instead of Eq. (14), the relation

$$d(A \wedge X) = dA \wedge X - A \wedge dX \tag{17}$$

can be used to write

$$\int_D X \wedge Y = \int_D d(A \wedge X) \cong \int_{\partial D} A \wedge X + \int_T A \wedge X \,. \tag{18}$$

Here, the arc C has been enclosed in a tube T and the symbol $\cong$ has been used to indicate that small elements of surface, which will vanish as the diameter of the tube is allowed to approach C in the limit, have been ignored. Combining Eqs. (16) and (18),

$$\int_{\partial D} A \wedge X \cong \int_{\partial D} \phi Y + \int_{\Gamma} X \int_{\Sigma} Y - \int_{T} A \wedge X \ . \quad (19)$$

Evaluating the last integral of the latter expression is most readily accomplished by transforming to classical notation. The integrand $Z \wedge X$ becomes $\mathbf{A}\times\mathbf{X}\cdot\hat{\mathbf{n}}\, dT$, where $d\mathbf{T}$ is the element of area on the tube oriented along the normal $\hat{\mathbf{n}}$.

Noting that the tube is the topological product of a circle, L (with tangent $\mathbf{dl}$) and the cut C (with tangent $\mathbf{dl}'$), the element of surface area can be written as $d\mathbf{T}=\mathbf{dl}\times\mathbf{dl}'$. The integrand is then

$$(\mathbf{A}\times\mathbf{X})\cdot(\mathbf{dl}\times\mathbf{dl}')=(\mathbf{A}\cdot\mathbf{dl})(\mathbf{X}\cdot\mathbf{dl}')-(\mathbf{A}\cdot\mathbf{dl}')(\mathbf{X}\cdot\mathbf{dl}) \ .$$

The space $\bar{D}$, interior to the toroid, is singly connected by virtue of the cut Σ, while the cut C connects the two components of $\bar{D}$. Then, by Stokes' theorem, the last term can be seen to vanish since $\nabla\times\mathbf{X}=0$. Consider the remaining two terms in the limit that $T\to C$. Stokes' theorem can again be used to transform the $\mathbf{A}\cdot\mathbf{dl}$ term, and because $\mathbf{X}$ approaches a value depending only on its limiting position on C, the integral can be written as the product

$$\int_{T}(\mathbf{A}\times\mathbf{X})\cdot\hat{\mathbf{n}}\, dT=\int_{\partial D}\mathbf{Y}\cdot\hat{\mathbf{n}}\, dS\int_{C}\mathbf{X}\cdot\mathbf{dl}' \ . \quad (20)$$

Thus, Eq. (19) can be written as

$$\int_{\partial D} A \wedge X = \int_{\partial D} \phi Y + \int_{\Gamma} X \int_{\Sigma} Y - \int_{\partial D} Y \int_{C} X \ . \quad (21)$$

The general form of Eq. (21), not yet written in terms of magnetic fields and potentials, can now be used in the case that ∂D is a magnetic surface. Let A now be identified with the vector potential 1-form. If the current vanishes outside the torus, an appropriate choice for X and Y is $X=*B$ and $Y=B$. These are consistent choices since $dX=d(*B)=\mu_0 *J=0$ and $dY=0$ in D. With the condition that the normal component of $\mathbf{B}$ vanish on ∂D, the open-surface integral $\int_\Sigma B$ is a period which can be interpreted as a flux, and Eq. (21) becomes

$$\int_{\partial D} A \wedge *B = \int_{\Gamma} *B \int_{\Sigma} B = \mu_0 I_\phi \Phi \ , \quad (22)$$

where I_ϕ is the azimuthal current inside the torus (outside the domain D) and Φ is the flux through the surface Σ. Since the current vanishes in the domain D, Stokes' theorem implies that

$$\int_{\partial D} A \wedge *B = \int_{D} B \wedge *B = \langle B,B\rangle \ . \quad (23)$$

Equation (22) can be used to interpret the boundary term of Eq. (11) in the case that the volume of integration is the interior of a torus. Attention, however, must be paid to the orientation of the boundary surface. To arrive at Eqs. (22) and (23), the domain D was chosen to be the exterior of the toroidal surface; on the other hand, the volume of integration in Eq. (11) is now assumed to be the interior of the torus. A surface element on the boundary thus has opposite orientation for the two cases, which will introduce a minus sign when substituting the value of the integral into Eq. (11).

Thus, not surprisingly, the total energy as expressed by Eq. (11) is the sum of the energy interior to the torus, where currents may be present, and that in the field exterior to the torus, represented by the boundary term. Since it has been assumed that there are no currents in the domain exterior to the torus, the field in this region must be due to currents within the torus. Currents outside S_0 do not contribute to the magnetic energy in the space bounded by S_0+S_1 because of the boundary conditions imposed and the fact that the space outside S_0 is simply connected. If the boundary configuration were different, so that the space outside S_0 was not simply connected (for example, if S_0 was a second torus so that S_1 was nested within it), then there could be additional contributions to the field energy between the toroids due to currents exterior to S_0.

PHYSICAL EXAMPLES

Ring current in a toroidal domain

Consider a current I_ϕ in a perfectly conducting toroidal domain D. The energy of the configuration can be written in terms of the currents in the domain as

$$E=\tfrac{1}{2}\int_{D} A \wedge *J \ . \quad (24)$$

For consistency, it is necessary that $\int_{\Gamma_1'} A=\Phi_{\Sigma_2}$ [see Fig. 2(b)]. The meaning of a gauge transformation $A'=A+d\chi$, with χ a multivalued function, will be discussed shortly. Using Stokes' theorem, Eq. (24) becomes

$$\mu_0 E=\tfrac{1}{2}\int_{D} B \wedge *B - \tfrac{1}{2}\int_{\partial D} A \wedge *B \ . \quad (25)$$

The first term is clearly the energy of the field within the toroidal conductor, while Eq. (22) and the discussion following Eq. (23) allow the boundary term to be written as

$$-\int_{\partial D} A \wedge *B = \int_{\Gamma_2} *B \int_{\Sigma_2} B = \mu_0 I_\phi \Phi_{\Sigma_2} \ . \quad (26)$$

To see this, note that the first integral in the product, using Stokes' theorem, is

$$\int_{\Gamma_2} *B = \int_{\Sigma_1} d(*B) = \mu_0 \int_{\Sigma_1} *J = \mu_0 I_\phi \ . \quad (27)$$

Thus, the energy is the sum of the field energy within the current carrying toroidal domain and the energy in the field exterior to the domain due to currents within D, the latter energy being represented by the boundary term.

A "multiturn coil" having n "turns" would have Γ_1' traversed n times. This is equivalent to performing a gauge transformation $A'=A+d\chi$ in Eq. (24). This can be seen by considering the $d\chi$ term, $\int_D d\chi \wedge *J$, in such a transformation. Because $d\chi$ is a closed 1-form and $*J$ a closed 2-form, the discussion preceding Eq. (16) allows this term to be written as

$$\int_{\Gamma_1} d\chi \int_{\Sigma_1} *J = I_\phi \int_{\Gamma_1'} d\chi \ .$$

[The fact that $\int_{\partial D}\chi(*J)=0$ has been used.] Now the jump in the function χ across Σ_1, $[\chi]=\int_{\Gamma_1'}d\chi$, must yield Φ_{Σ_2} for each "turn" of the coil. Thus, the energy in the field exterior to the toroid would be $nI_\phi\Phi_{\Sigma_2}$. A gauge transformation corresponds to multiplying the energy in the field by some integer n. If one constructs the simply connected, universal covering space $\mathbb{D}$ of the domain D interior to the torus, n is essentially the winding number, uniquely determined by the path homotopy class of Γ_1'. In general, gauge transformations in $\mathbb{D}$ affect the physics in D.

Note that because the normal component of **B** vanishes on ∂D, $\int_{\Sigma_2}B$ is an open-surface period. The second of de Rham's theorems then guarantees the existence of a B exterior to D such that

$$\int_{\Sigma_2}B=\int_{\Gamma_1'}A=\Phi_{\Sigma_2}\ . \tag{28}$$

However, while the existence of B is assured, its uniqueness is not; in this connection, see the comments at the end of the last section.

Application to force-free fields

Perhaps the most well-known solution to the cylindrically symmetric, force-free magnetic-field equations with constant α is the Lundquist [28] solution given by

$$\mathbf{H}=A_0J_1(\alpha r)\hat{\boldsymbol{\phi}}+A_0J_0(\alpha r)\hat{\mathbf{z}}\ , \tag{29}$$

where A_0 is an arbitrary constant.

If one chooses to apply this solution in a cylindrical region of radius a, such that $J_0(\alpha a)=0$, this solution matches smoothly—no surface currents are required—to an external field given by $\{0,(aA_0/r)J_1(\alpha a),0\}$, which only has a nonvanishing $\hat{\boldsymbol{\phi}}$ component. Since there is a nonvanishing field outside the cylindrical boundary at $r=a$, it is clear that the surface integral of the last section cannot vanish. Nevertheless, consider the following *incorrect* argument: Since α is a constant, $\nabla\times\mathbf{B}=\alpha\mathbf{B}$ can be written as $\nabla\times(\nabla\times\mathbf{A}-\alpha\mathbf{A})=\mathbf{0}$. This means that $\mathbf{B}=\alpha\mathbf{A}$ and the surface integral vanishes since $\mathbf{H}\times\mathbf{A}=\mathbf{0}$.

Why is this argument incorrect? There are two reasons: (1) Unless $z=-\infty$ and $z=+\infty$ are identified, there must be additional return currents outside the cylindrical domain. (2) Assume that henceforth this identification is made: The issue of return currents is eliminated but the solution will have the topology of a torus [29]. Since the equation $\nabla\times(\nabla\times\mathbf{A}-\alpha\mathbf{A})=\mathbf{0}$ actually implies that $\mathbf{B}=\alpha\mathbf{A}+\nabla\chi$, the statement that $\mathbf{H}\times\mathbf{A}=\mathbf{0}$ is not gauge invariant because χ is a multivalued function in such a multiply connected domain [30].

The force-free relation $d(*B)=\alpha\,dA$, with α being a constant, implies that $*B=\alpha A+d\chi$. The boundary term is then given by (see Fig. 2)

$$\int_{\partial D}A\wedge *B=\int_{\partial D}A\wedge d\chi=\int_D d(\chi B)$$
$$=\int_{\partial D}\chi B+\int_{\Sigma_1}[\chi]B\ , \tag{30}$$

where the fact that

$$d(A\wedge d\chi)=B\wedge d\chi=d(\chi B)$$

has been used. The first term on the right-hand side of Eq. (30) vanishes and, by the argument following Eq. (15), $[\chi]$ can be taken out of the second integral (note the comment on finite integrals in [29]). Thus,

$$\int_{\partial D}A\wedge *B=\int_{\Gamma_1}d\chi\int_{\Sigma_1}B\ . \tag{31}$$

Using the force-free relation, the integral over Σ_1 becomes

$$\int_{\Sigma_1}B=\frac{1}{\alpha}\int_{\Sigma_1}d(*B)=\mu_0\frac{I_\phi}{\alpha}\ , \tag{32}$$

where I_ϕ is the azimuthal current. Since $[\chi]$ is constant on Σ_1, Γ_1 may be chosen in ∂D, bounding Σ_2 (shown as Γ_1' in Fig. 2). Choose A' such that $\int_{\Gamma_1'}A'=\Phi_{\Sigma_2}$. Then in the interior domain D, since $*B=\alpha A+d\chi$, the objective will be to find $d\chi$ such that $\int_{\Gamma_1'}A=\int_{\Gamma_1'}A'$. Now, $A=(1/\alpha)(*B-d\chi)$, so that

$$\int_{\Gamma_1'}A=\frac{1}{\alpha}\int_{\Gamma_1'}*B-\frac{1}{\alpha}\int_{\Gamma_1'}d\chi\ . \tag{33}$$

The first term on the right-hand side vanishes since

$$\int_{\Gamma_1'}*B=\mu_0\int_{\Sigma_2}*J=0\ .$$

Thus, the requirement that $\int_{\Gamma_1'}A=\int_{\Gamma_1'}A'$ implies, with the choice $\chi\to-\chi$, that

$$\int_{\Gamma_1'}d\chi=\alpha\Phi_{\Sigma_2}\ . \tag{34}$$

Equations (32) and (34) then give for the boundary term,

$$\int_{\partial D}A\wedge *B=\mu_0\Phi_{\Sigma_2}I_\phi\ , \tag{35}$$

which will vanish only if the flux through Σ_2 vanishes, i.e., $I_\phi=0$. Thus, the interpretation of this term for force-free fields is consistent with that given in the previous section: it represents the energy in the field exterior to ∂D.

For the Lundquist solution, the component of **B** normal to ∂D vanishes, implying that A is a local surface gradient [31], $A=d_s\chi$. Since A is a closed 1-form on ∂D, by de Rham's first theorem, A will be exact (χ will be single valued) if $\int_{\mathbf{z}_i}A=0$, where $\mathbf{z}_i$ are the independent 1-cycles on ∂D, i.e., if $\int_{\Gamma_1'}A=\int_{\Gamma_2}A=0$, or $\int_{\Sigma_2}B=\int_{\Sigma_1}B=0$. This condition is equivalent to $I_\phi=0$, and can be met with a nonvanishing field in the interior of the torus if αr_0, where r_0 is the minor radius of the torus, is a zero of $J_1(\alpha r)$. This can be seen from

$$I_\phi=2\pi\frac{A_0}{\alpha}\int_0^{\rho_0}J_0(\rho)\rho\,d\rho=2\pi\frac{A_0}{\alpha}\rho_0J_1(\rho_0),\quad \rho=\alpha r \tag{36}$$

or by simply noting that the $\hat{\boldsymbol{\phi}}$ component of the field given by Eq. (29) has a zero at $J_1(\alpha r)=0$ and that by

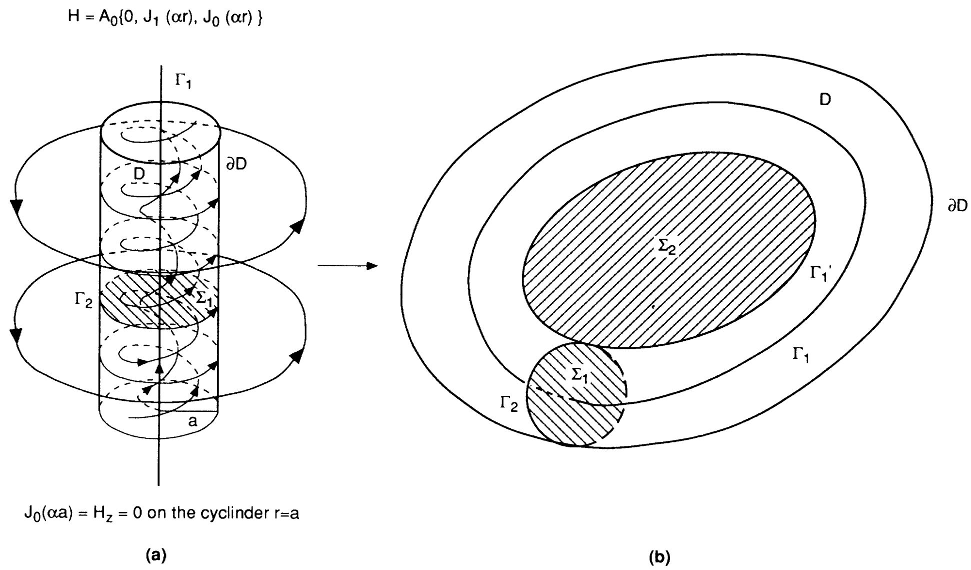

FIG. 2. The Lundquist solution can be given the topology of a torus by identifying $z=\pm\infty$ (finite integrals on Γ_1 and Σ_2 can be had by identifying $z=\pm l$ rather than $z=\pm\infty$). The domain D where the force-free condition holds is interior to the torus.

Ampere's circuital law, the current density integrated over the enclosed area must vanish.

SUMMARY

In general, the energy contained in force-free magnetic fields is given by the right-hand side of Eq. (11). In the case that α is a constant, the energy is the product of α and the helicity. Whether α is constant or not, when the field forms toroidal magnetic surfaces, the energy can be written as the sum of the usual expression for magnetic energy and a generally nonvanishing boundary term $\int_{\partial V} A \wedge *B$. Physically, for a toroidal domain V whose exterior is current free, this term represents the energy in the magnetic field exterior to V due to currents within the domain. The boundary term can vanish for toroidal force-free magnetic fields even though the field is nonzero in V.

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[23] A cycle is a chain whose boundary vanishes, while the n-chain is a formal sum, with constant coefficients, of n-simplices.

[24] The domain within which a closed p-form will be exact depends on p: if $p=0$, the domain must be connected; if $p=1$, the domain must be simply connected; and if $p=2$, one must be able to shrink any spherical surface to a point.

[25] In a simply connected domain, where the current vanishes so that **B** is harmonic ($\nabla^2\mathbf{B}=0$), if the normal component of **B** vanishes on the boundary, **B** will be identically zero.

[26] See A. A. Blank, K. O. Friedrichs, and H. Grad, Ref. [18].

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[31] Orient the normal to the surface along the $\hat{\mathbf{z}}$ direction. Since the normal component of **B** vanishes, $B_z=0$, and $\partial_x A_y - \partial_y A_x = 0$. A can therefore be represented as a "local surface gradient" in the sense that $A_y=\partial_y\chi$ and $A_x=\partial_x\chi$.