Axially symmetric solutions to the force-free magnetic-field equations in spherical and cylindrical coordinates

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The force-free magnetic-field condition $\nabla\times\mathbf{H}=\alpha\mathbf{H}$ is expressed in terms of a flux function Ψ; α is then also a function of Ψ, and with suitable restrictions the resulting equations can be separated and solved. The case of spherical coordinates yields four sets of solutions that are shown to be dependent and equivalent to a simple generalization of those given by Chandrasekhar [Proc. Natl. Acad. Sci. U.S.A. **42**, 1 (1956)]. Similarly, the case of cylindrical coordinates results in a generalization of the solution given by Furth, Levine, and Waniek [Rev. Sci. Instrum. **28**, 949 (1957)].

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INTRODUCTION

The foundation for much of the work on force-free magnetic fields can be found in the literature of fluid mechanics. There, the analog to the force-free magnetic-field condition appears as $\nabla\times\mathbf{V}=\Omega\mathbf{V}$, where $\mathbf{V}$ is the fluid velocity. Solutions to this equation, where Ω is a function of position, are known as Beltrami fields; solutions for the case where Ω is a constant are known as Trkalian fields. Bjorgum [1,2] has written two comprehensive papers on these types of fluid flow and has also given a method of finding Trkalian fields from solutions to the scalar wave equation. This is essentially the same method later developed by Chandrasekhar and Kendall [3] for application to force-free magnetic fields. A detailed exposition of this material is given in the thesis of Buck [4].

When α, in the force-free magnetic-field condition $\nabla\times\mathbf{H}=\alpha\mathbf{H}$, is not a constant, the technique of finding solutions to this equation by solving the scalar wave equation is generally not applicable. An alternative approach is to express the magnetic-field components in terms of a flux function Ψ (comparable to the stream function in hydrodynamics) and then demand that the field satisfy the force-free condition. The equation that results is a known form of the Grad-Shafranov equation [5,6] for constant fluid pressure. The latter is a second-order, nonlinear potential equation containing a single arbitrary function.

The Grad-Shafranov equation can then be separated in terms of a second arbitrary function $g(\Psi)$. This results in a form of Bernoulli's equation for the function α and an equation for the flux function Ψ, which may be solved for the case where $g(\Psi)$ takes a particularly simple form. The solutions to these equations, in both spherical and cylindrical coordinates, are generalizations of the solutions given by Chandrasekhar [7], Furth, Levine, and Wanick [8], and Emets and Zamidra [9].

I. MAGNETIC-FIELD COMPONENTS IN TERMS OF THE FLUX FUNCTION

When the magnetic field is symmetric about an axis, one may introduce the analog of Stokes's stream function for incompressible, axially symmetric velocity fields. Such a function is known as the flux function [10], and is introduced as follows: In arbitrary curvilinear coordinates, the divergence of $\mathbf{B}$ is given by

$$\nabla\cdot\mathbf{B}=\frac{1}{\sqrt{g}}\partial_{x_i}(\sqrt{g}B^i) \tag{1}$$

where $g=|g_{ij}|$, $i=1,2,3$ and the symbols have their conventional meanings. If $\mathbf{B}$ is independent of one coordinate (say x_3) the equation

$$\nabla\cdot\mathbf{B}=\frac{1}{\sqrt{g}}\partial_{x_1}(\sqrt{g}B^1)+\frac{1}{\sqrt{g}}\partial_{x_2}(\sqrt{g}B^2)=0 \tag{2}$$

is obviously satisfied if one introduces a function Ψ, such that

$$\sqrt{g}B^1=\partial_{x_2}\psi\ ,\quad \sqrt{g}B^2=-\partial_{x_1}\psi\ . \tag{3}$$

For orthogonal coordinates, $g^{ik}=g_{ik}=0$ for $i\neq k$; $g_{ii}=h_i^2$ and $g^{ii}=1/h_i^2$; $\sqrt{g}=h_1h_2h_3$. The physical components [11] are then h_iB^i. From Eq. (3) these can be written

$$B_{(1)}=\frac{h_1}{\sqrt{g}}\partial_{x_2}\psi=\frac{1}{h_2h_3}\partial_{x_2}\psi\ ,$$
$$B_{(2)}=-\frac{h_2}{\sqrt{g}}\partial_{x_1}\psi=-\frac{1}{h_1h_3}\partial_{x_1}\psi \tag{4}$$

where the subscript in parentheses designates the physical component (the parentheses will henceforth be dropped).

In spherical coordinates, $(x_1,x_2,x_3)\rightarrow(r,\theta,\phi)$ and $h_1=1$, $h_2=r$, $h_3=r\sin\theta$. Thus, assuming axial symmetry so that the fields are independent of ϕ,

$$B_r=\frac{1}{r^2\sin\theta}\partial_\theta\psi \ , \quad B_\theta=-\frac{1}{r\sin\theta}\partial_r\psi \ . \tag{5}$$

In cylindrical coordinates, $(x_1,x_2,x_3)\to(r,\phi,z)$ and $h_1=h_3=1$, $h_2=r$. Consequently, for axial symmetry,

$$B_r=-\frac{1}{r}\partial_z\psi \ , \quad B_z=\frac{1}{r}\partial_r\psi \ . \tag{6}$$

It is interesting to note that the introduction of a flux function Ψ is equivalent to writing the field in terms of two functions P and T that, respectively, generate the poloidal and toroidal components of the field. In spherical coordinates, for example, expanding

$$\mathbf{H}=\hat{\mathbf{z}}\times\mathbf{r}T(r,\theta)+\nabla\times[\hat{\mathbf{z}}\times\mathbf{r}P(r,\theta)] \tag{7}$$

results in

$$\mathbf{H}=\frac{1}{r^2\sin\theta}\partial_\theta[r^2\sin\theta P(r,\theta)]\hat{\mathbf{r}} -\frac{1}{r\sin\theta}\partial_r[r^2\sin\theta P(r,\theta)]\hat{\theta}+rT(r,\theta)\hat{\phi} \ . \tag{8}$$

Defining the flux function as $r^2\sin\theta P(r,\theta)$ gives the same result as Eq. (5). In this formulation B_ϕ is given by rT.

II. FORCE-FREE CONDITION IN SPHERICAL COORDINATES

Substituting the magnetic-field components, in the form given by Eq. (5), into the force-free condition $\nabla\times\mathbf{B}=\alpha\mathbf{B}$ results in three equations,

$$\frac{\alpha}{r\sin\theta}\partial_\theta\psi=(\partial_\theta B_\phi+B_\phi\cot\theta) \ , \tag{9}$$

$$\frac{\alpha}{r\sin\theta}\partial_r\psi=(\partial_r B_\phi+B_\phi/r) \ , \tag{10}$$

$$\alpha B_\phi=-\frac{1}{r\sin\theta}\Delta^*\psi \tag{11}$$

where $B_\phi=B_\phi(r,\theta)$ and

$$\Delta^*\psi=\partial_r^2\psi+\frac{1}{r^2}\partial_\theta^2\psi-\frac{1}{r^2}\cot\theta\partial_\theta\psi \ . \tag{12}$$

Equations (9) and (10) together imply that

$$\partial_r\psi\partial_\theta(r\sin\theta B_\phi)-\partial_\theta\psi\partial_r(r\sin\theta B_\phi)=0 \tag{13}$$

which will be satisfied if $r\sin\theta B_\phi=f(\psi)$, where $f(\Psi)$ is an arbitrary function of Ψ alone; thus

$$B_\phi=\frac{1}{r\sin\theta}f(\psi) \ . \tag{14}$$

Equation (11) may therefore be written as $\Delta^*\psi=-\alpha f(\psi)$. α may now be determined from either Eq. (9) or Eq. (10) as $\alpha=f'(\Psi)$, where the prime indicates differentiation by Ψ. Defining $\mu=\cos\theta$, Eq. (11) may be written in final form as

$$\partial_r^2\psi+\frac{(1-\mu^2)}{r^2}\partial_\mu^2\psi=-f(\psi)f'(\psi) \ . \tag{15}$$

As mentioned in the Introduction, this is a form of the Grad-Shafranov equation. Approximate nonconstant α solutions to this equation have been discussed in the astrophysical literature [12,13]. Derivations are given in the following sections of exact nonconstant α solutions that are then shown to be generalizations of known solutions.

III. SEPARATION AND SOLUTION OF THE GRAD-SHAFRANOV EQUATION IN TERMS OF AN ARBITRARY FUNCTION

Since $f(\Psi)$ is an arbitrary function of Ψ, the right-hand side of Eq. (15) can be written as $g(\Psi)=-f(\Psi)\alpha$, where $g(\Psi)$ is a second arbitrary function of Ψ. The purpose of this definition is to allow α to be determined in terms of $g(\Psi)$. Differentiating $g(\Psi)=-f(\Psi)\alpha$ with respect to Ψ, one obtains

$$\alpha'-\frac{g'}{g}\alpha-\frac{1}{g}\alpha^3=0 \ . \tag{16}$$

This is a form of Bernoulli's equation and is completely integrable. Letting $\xi=\alpha^{-2}$, the solution is

$$\xi=A\exp\left[-2\int^\psi\frac{g'}{g}d\psi\right]-2\exp\left[-2\int^\psi\frac{g'}{g}d\psi\right]\times\int^\psi\frac{1}{g}\exp\left[2\int^\psi\frac{g'}{g}d\psi\right]d\psi \tag{17}$$

where A is a constant of integration.

While $g(\Psi)$ is still arbitrary in Eq. (17), the remaining equation, $\Delta^*\psi=g(\psi)$, will apparently only separate if $g(\Psi)=-\kappa^2\Psi$, where κ is a constant. If $g(\Psi)$ is so restricted and Ψ is assumed to be of the form $\psi=\Phi(r)\Gamma(\mu)$, the equation separates, in terms of a separation constant λ, into

$$\Phi''(r)+\left[\kappa^2-\frac{\lambda}{r^2}\right]\Phi(r)=0 \tag{18}$$

and

$$\Gamma''(\mu)+\frac{\lambda}{1-\mu^2}\Gamma(\mu)=0 \ . \tag{19}$$

For $g(\Psi)=-\kappa^2\Psi$, Eq. (17) gives $\xi=A\Psi^{-2}+1/\kappa^2$ so that

$$\alpha=\frac{\kappa\psi}{(B^2+\psi^2)^{1/2}} \ , \quad B^2=A\kappa^2 \ . \tag{20}$$

Note that if $B=0$, $\alpha=\kappa$; and since $\alpha=f'(\Psi)$, apart from a constant of integration, $f(\Psi)$ is determined by Eq. (20) to be

$$f(\psi)=\int\alpha\, d\psi=\kappa(B^2+\psi^2)^{1/2} \ . \tag{21}$$

Now if $f(\Psi)$ does not have a zero on the polar axis, Eq. (14) implies that B_ϕ is singular. Therefore, if the constant B is not required to vanish, the axis must be excluded from the domain [14]. This is a physically reasonable requirement if B is interpreted as being proportional to an axial line current. Because of the symmetry of the problem, such an axial current will only generate a toroidal field, and can flow without altering the force-free character of the configuration. While such a line current is mathematically acceptable, real world applications would require the imposition of boundary conditions corresponding to a finite current density.

Consider Eq. (18). If λ is restricted to the values $\lambda=n(n+1)$, $n=0,\pm1,\pm2,\ldots$, the solutions to this equation are the Riccati-Bessel functions with argument κr; i.e., $(\kappa r)c_n(\kappa r)$ where $c_n(\kappa r)$ is one of $\{j_n(\kappa r), y_n(\kappa r), h_n^{(1)}(\kappa r), h_n^{(2)}(\kappa r)\}$. The properties of these functions follows from those of the spherical Bessel functions.

The solution to Eq. (19) can be given in terms of the generalized Jacobi polynomials [15],

$$(1-\mu)^{(\alpha+1)/2}(1-\mu)^{(\beta+1)/2}P_m^{(\alpha,\beta)}(\mu)\ ,\qquad \alpha,\beta=\pm1\ ,\ m=0,1,2,\ldots \tag{22}$$

provided Eq. (19) is rewritten as

$$\Gamma''(\mu)+G^{(\alpha,\beta)}(m,\mu)\Gamma(\mu)=0 \tag{23}$$

and $G^{(\alpha,\beta)}(m,\mu)$ is restricted to the form

β \ α	$+1$	-1
$+1$	$\frac{m(m+3)+2}{(1-\mu^2)}$	$\frac{m(m+1)}{(1-\mu^2)}$
-1	$\frac{m(m+1)}{(1-\mu^2)}$	$\frac{m(m-1)}{(1-\mu^2)}$

$G^{(1,-1)}$ and $G^{(-1,1)}$ will be compatible with the requirement from Eq. (18) that $\lambda=n(n+1)$ if $m=n$. The same is true for $G^{(1,1)}$ if $m=n-1$ and $G^{(-1,-1)}$ if $m=n+1$. (While other possibilities exist for negative n, they add nothing substantive to the discussion.)

Given these constraints, there are apparently four possible solutions given by Eq. (22):

$$\Gamma(\mu)=\left\{\begin{array}{c}(1-\mu)P_n^{(1,-1)}(\mu)\\ (1+\mu)P_n^{(-1,1)}(\mu)\\ (1-\mu)(1+\mu)P_{n-1}^{(1,1)}(\mu)\\ P_{n+1}^{(-1,-1)}(\mu)\end{array}\right\}\ ,\quad n\geq1\ . \tag{24}$$

The restriction to $n\geq1$ is imposed to exclude the $n=0$ case for the first two solutions because these can be ruled out on physical grounds (they require sources and sinks on the negative z axis); and for $n=0$, the third and fourth sets of solutions are zero. In the case of the fourth solution, $P_1^{(-1,-1)}(\mu)=0$ since this polynomial satisfies the condition $n+\alpha+\beta+k=0$, $\alpha=-j$, $1\leq k\leq j\leq n$ (see Ref. [15]). Note also that all four possible solutions have zeros at $\mu=\pm1$, corresponding to the positive and negative z axis, respectively. Each of the solutions of Eq. (24) can be combined with a solution of Eq. (18) of order n to form the flux function.

The four solutions given by Eqs. (24) cannot, however, be linearly independent since Eq. (23) is only of second order. It is shown in the Appendix that the following relations hold for $n\geq1$:

$$(1-\mu)P_n^{(1,-1)}(\mu)=-(1+\mu)P_n^{(-1,1)}(\mu)\ , \tag{25}$$

$$P_{n+1}^{(-1,-1)}(\mu)=\tfrac{1}{4}(\mu+1)(\mu-1)P_{n-1}^{(1,1)}(\mu)\ , \tag{26}$$

$$P_{n+1}^{(-1,-1)}(\mu)=\frac{n}{2(n+1)}(\mu+1)P_n^{(-1,1)}(\mu)\ , \tag{27}$$

$$(1+\mu)P_n^{(-1,1)}(\mu)=-\frac{(n+1)}{2n}(1-\mu)(1+\mu)P_{n-1}^{(1,1)}(\mu)\ , \tag{28}$$

$$(1-\mu)P_n^{(1,-1)}(\mu)=-\frac{2(n+1)}{n}P_{n+1}^{(-1,-1)}(\mu)\ , \tag{29}$$

$$(1-\mu)P_n^{(1,-1)}(\mu)=\frac{(n+1)}{2n}(1-\mu)(1+\mu)P_{n-1}^{(1,1)}(\mu)\ . \tag{30}$$

This set of relations allows any of the four solutions to be written in terms of any other, and in particular the third. The latter solution may in turn be easily related to those given by Chandrasekhar (Ref. [7]): Let $B=0$ so that $\alpha=\kappa$. Remembering that $G^{(1,1)}(m,\mu)$ will be compatible with $c_n(\kappa r)$ if $m=n-1$, the flux function associated with $G^{(1,1)}(m,\mu)$ is given by

$$\psi=(\alpha r)c_m(\alpha r)(1+\mu)(1-\mu)P_m^{(1,1)}(\mu)\ . \tag{31}$$

The discussion at the end of Sec. I and the definition of P following Eq. (8), combined with the relation

$$C_n^{\beta}(\mu)=\frac{\Gamma(\beta+\frac{1}{2})\Gamma(2\beta+n)}{\Gamma(2\beta)\Gamma(\beta+n+\frac{1}{2})}P_n^{(\beta-1/2,\beta-1/2)}(\mu)\ ,\quad \beta\neq0 \tag{32}$$

allow this set of solutions to be written in terms of

$$P^m=\frac{D_{m+3/2}(\alpha r)}{r^{3/2}}C_m^{3/2}(\mu)\ . \tag{33}$$

Here $D_{m+3/2}(\alpha r)$ is a general cylinder function of order $n+\frac{3}{2}$ and the $C_m^{3/2}(\mu)$ are the Gegenbauer or ultraspherical polynomials. The function P that generates the poloidal field is given by $P=\sin\theta P^m$. For $B\neq0$, the four sets of solutions given in Eq. (24) are therefore seen to be a simple generalization of Chandrasekhar's results.

Other solutions appearing in the literature can also be related to those given here. For example, consider the case where $\alpha=+1$, $\beta=-1$. (As has been shown, the particular case chosen is irrelevant.) Then the flux function is, apart from an arbitrary multiplicative constant

$$\psi=(\kappa r)c_n(\kappa r)(1-\mu)P_n^{(1,-1)}(\mu)\ . \tag{34}$$

The magnetic-field components are then given by Eqs. (5), (14), and (21) as

$$B_r=\frac{\kappa}{r}c_n(\kappa r)\left\{\frac{(n+1)}{n(1+\mu)}[\mu nP_n^{(1,-1)}(\mu)-(n-1)P_{n-1}^{(1,-1)}(\mu)]\right\}\ , \tag{35}$$

$$B_\theta = \frac{\kappa(1-\mu)}{r\sin\theta} P_n^{(1,-1)}(\mu)[nc_n(\kappa r)-(\kappa r)c_{n-1}(\kappa r)] \ , \quad (36)$$

$$B_\phi = \frac{\kappa}{r\sin\theta}\{B^2+(\kappa r)^2[c_n(\kappa r)]^2(1-\mu)^2 \times [P_n^{(1,-1)}(\mu)]^2\}^{1/2} \ . \quad (37)$$

For $c_n(\kappa r)=j_n(\kappa r)$, and $n=1$, the expression for ψ becomes

$$\psi = \left[\frac{\sin\kappa r}{\kappa r} - \cos\kappa r\right]\sin^2\theta \ . \quad (38)$$

This is the form of the flux function given by Morikawa [16] and also, without the imposition of the boundary condition of a uniform external field, that given by Emets and Zamidra (Ref. [9]). Both Morikawa and Emets and Zamidra only give the lowest-order ($n=1$) solution. Morikawa imposes the boundary condition of a perfectly conducting spherical cavity and assumes that α is constant, while Emets and Zamidra show that the constant B in Eq. (20) can be interpreted to be proportional to an axial current. The solutions of Eqs. (35)–(37) are then a generalization of the particular solutions given by these authors.

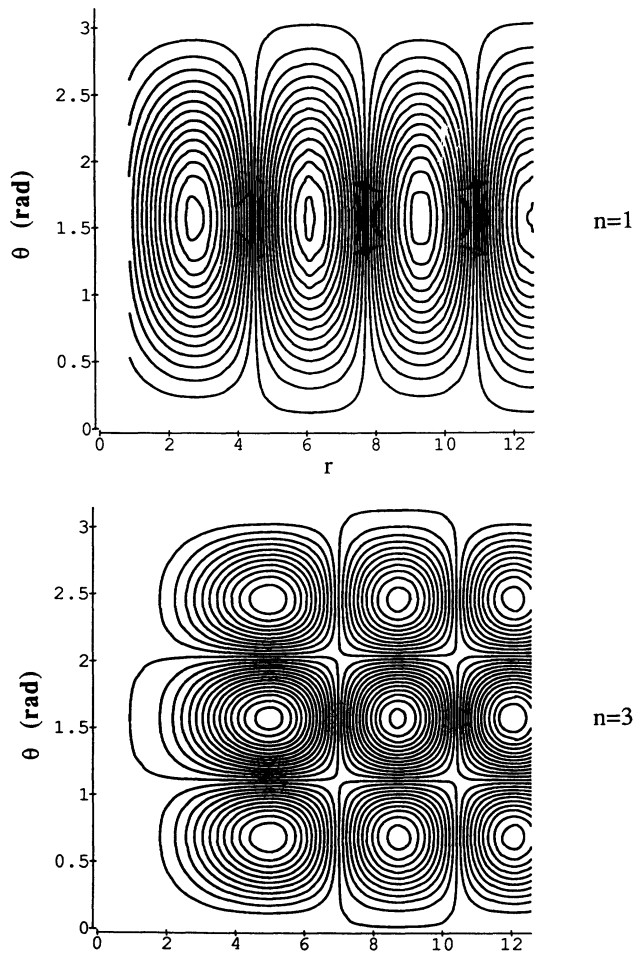

FIG. 1. Contour plot of

$$\psi = r\sqrt{\pi/2r}\,J_{n+1/2}(r)(1-\mu)P_n^{(1,-1)}(\mu)$$

for $n=1$ and 3.

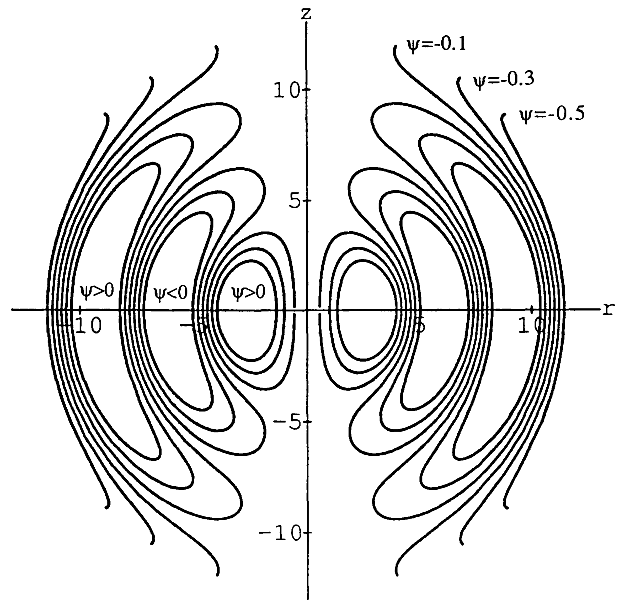

FIG. 2. Parametric plot of

$$\psi = r\sqrt{\pi/2r}\,J_{n+1/2}(r)(1-\mu)P_n^{(1,-1)}(\mu)$$

for $n=1$ as a function of r and $z=r\cos\theta$. In the figure the contours are for $\psi=\pm0.1, \pm0.3, \pm0.5$. Note that $\psi=0$ on the z axis.

The character of these solutions is illustrated in Fig. 1 for $n=1,3$. There, the surfaces of constant ψ are plotted as a function of r and θ (ϕ being assumed constant). Irregularities in the plot should be ignored since they merely reflect limitations in the way the function was sampled for plotting. Figure 2 shows the $n=1$ case for particular values of ψ as a function of $z=r\cos\theta$ and r. The magnetic surfaces, outside the familiar central one, are toroids with a bananalike cross section.

IV. SOLUTION IN CYLINDRICAL COORDINATES

Using the definitions of the field components in Eq. (6), and following the same procedures as in the case of spherical coordinates results in the three equations

$$\partial_r^2\psi - \frac{1}{r}\partial_r\psi + \partial_z^2\psi = -f(\psi)f'(\psi) \ , \quad (39)$$

$$B_\phi = \frac{1}{r}f(\psi) \ , \quad (40)$$

$$\alpha = f'(\psi) \ . \quad (41)$$

If $g(\psi)$ is defined as $g(\psi)=-f(\psi)f'(\psi)$, α will again satisfy Bernoulli's equation with the solution given by Eq. (17).

As before, Eq. (39) will apparently only separate if $g(\psi)=-\kappa^2\psi$, yielding

$$\Phi''(r) - \frac{1}{r}\Phi'(r) + (\kappa^2-\lambda^2)\Phi(r) = 0 \ , \quad (42)$$

$$\Gamma''(z) + \lambda^2\Gamma(z) = 0 \quad (43)$$

where here $\psi=\Phi(r)\Gamma(z)$ and λ^2 is again a separate constant. Both equations are immediately solvable, and apart from an arbitrary constant of integration, have the solutions

$$\Phi(r)=rD_1[(\kappa^2-\lambda^2)^{1/2}r] \quad (44)$$

and

$$\Gamma(z)=\begin{Bmatrix}\sin\lambda z\\ \cos\lambda z\end{Bmatrix} \quad (45)$$

where D_1 is a cylinder function of first order. Here, while the constants of integration and the particular cylinder function may be chosen to fit the boundary conditions of the problem at hand, the order of the cylinder function is fixed.

Defining $\gamma=(\kappa^2-\lambda^2)^{1/2}$, Eqs. (6) and (40) give the field components as

$$B_r=\lambda D_1(\gamma r)\begin{Bmatrix}-\cos\lambda z\\ \sin\lambda z\end{Bmatrix}, \quad (46)$$

$$B_z=\frac{1}{r}[D_1(\gamma r)+(r\gamma)\partial_{(\gamma r)}D_1(\gamma r)]\begin{Bmatrix}\sin\lambda z\\ \cos\lambda z\end{Bmatrix}, \quad (47)$$

$$B_\phi=\frac{\kappa}{r}\left[B^2+r^2[D_1(\gamma r)]^2\begin{Bmatrix}\sin\lambda z\\ \cos\lambda z\end{Bmatrix}^2\right]^{1/2}. \quad (48)$$

For the particular case $D_1(\gamma r)=J_1(\gamma r)$, $B=0$, and the choice of $\cos\lambda z$ in the expression for ψ, this is the "square toroid" solution given by Furth, Levine, and Waniek (Ref. [8]).

APPENDIX

To prove Eq. (25), consider the general relation

$$(1-\mu)P_n^{(\alpha+1,\beta)}(\mu)+(1+\mu)P_n^{(\alpha,\beta+1)}(\mu)=2P_n^{(\alpha,\beta)}(\mu)\,. \quad (A1)$$

Setting $\alpha=0$, $\beta=-1$ and $\alpha=-1$, $\beta=0$ results in two equations which, when added, yield

$$(1-\mu)P_n^{(1,-1)}(\mu)+(1+\mu)P_n^{(-1,1)}(\mu)+2P_n^{(0,0)}(\mu)-2[P_n^{(0,-1)}(\mu)+P_n^{(-1,0)}(\mu)]=0 \quad (A2)$$

where $P_n^{(0,0)}(\mu)$ are the Legendre polynomials $P_n(\mu)$. To show that the last term in brackets is equivalent to $P_n(\mu)$, consider the two relations [17]

$$(\alpha+\beta+2n)P_n^{(\alpha,\beta-1)}(\mu)=(\alpha+\beta+n)P_n^{(\alpha,\beta)}(\mu)+(\alpha+n)P_{n-1}^{(\alpha,\beta)}(\mu) \quad (A3)$$

and

$$(\alpha+\beta+2n)P_n^{(\alpha-1,\beta)}(\mu)=(\alpha+\beta+n)P_n^{(\alpha,\beta)}(\mu)-(\beta+n)P_{n-1}^{(\alpha,\beta)}(\mu)\,. \quad (A4)$$

Setting $\alpha=\beta=0$ results in two relations which, when combined, give the desired result $P_n^{(0,-1)}(\mu)+P_n^{(-1,0)}(\mu)=P_n^{(0,0)}(\mu)=P_n(\mu)$, which proves Eq. (25).

In Eq. (26), the polynomial $P_{n+1}^{(-1,-1)}(\mu)$ satisfies the differential equation,

$$D^2P_{n+1}^{(-1,-1)}(\mu)+\frac{n(n+1)}{1-\mu^2}P_{n+1}^{(-1,-1)}(\mu)=0 \quad (A5)$$

where $D^n=d^n/d\mu^n$. Now in general,

$$D^kP_n^{(\alpha,\beta)}(\mu)=2^{-k}(1+\alpha+\beta+n)_kP_{n-k}^{(\alpha+k,\beta+k)}(\mu)\,,\quad 0<k\le n \quad (A6)$$

where

$$(1+\alpha+\beta+n)_k=(1+\alpha+\beta+n)(2+\alpha+\beta+n)\cdots(1+\alpha+\beta+n+k-1)\,.$$

If, in the last equation, $\alpha=\beta=-1$, $k=2$, and n is set equal to $n+1$, one obtains

$$D^2P_{n+1}^{(-1,-1)}(\mu)=\tfrac{1}{4}n(n+1)P_{n-1}^{(1,1)}(\mu)\,. \quad (A7)$$

Combining this result with the differential equation, Eq. (A5), proves Eq. (26).

The proof of Eq. (27) uses the two relations

$$\tfrac{1}{2}(2n+\alpha+\beta+2)(1-\mu)P_n^{(\alpha+1,\beta)}(\mu)=(n+\alpha+1)P_n^{(\alpha,\beta)}(\mu)-(n+1)P_{n+1}^{(\alpha,\beta)}(\mu) \quad (A8)$$

and

$$\tfrac{1}{2}(2n+\alpha+\beta+2)(1+\mu)P_n^{(\alpha,\beta+1)}(\mu)=(n+\beta+1)P_n^{(\alpha,\beta)}(\mu)+(n+1)P_{n+1}^{(\alpha,\beta)}(\mu) \quad (A9)$$

with $\alpha=\beta=-1$ to obtain the two equations

$$n(1-\mu)P_n^{(0,-1)}(\mu)=nP_n^{(-1,-1)}(\mu)-(n+1)P_{n+1}^{(-1,-1)}(\mu) \quad (A10)$$

and

$$n(1+\mu)P_n^{(-1,0)}(\mu)=nP_n^{(-1,-1)}(\mu)+(n+1)P_{n+1}^{(-1,-1)}(\mu)\,. \quad (A11)$$

Subtracting Eq. (A10) from Eq. (A11),

$$P_{n+1}^{(-1,-1)}(\mu)=\frac{n}{2(n+1)}[(1+\mu)P_n^{(-1,0)}(\mu)-(1-\mu)P_n^{(0,-1)}(\mu)]\,. \quad (A12)$$

Consider the bracketed term on the right-hand side of Eq. (A12). By setting $\alpha=-1$, $\beta=0$ and $\alpha=0$, $\beta=-1$ in Eq. (A1), two equations are obtained which, when subtracted, result in

$$P_n^{(-1,0)}(\mu)-P_n^{(0,-1)}(\mu)=\tfrac{1}{2}[-2\mu P_n^{(0,0)}(\mu)+(1+\mu)P_n^{(-1,1)}(\mu)-(1-\mu)P_n^{(1,-1)}(\mu)] \ . \quad \text{(A13)}$$

The bracketed term of Eq. (A12) can be expanded and combined with Eq. (A13) to give

$$P_{n+1}^{(-1,-1)}(\mu)=\frac{n}{2(n+1)}\tfrac{1}{2}[(1+\mu)P_n^{(-1,1)}(\mu)-(1-\mu)P_n^{(1,-1)}(\mu)] \ . \quad \text{(A14)}$$

Equation (27) is then obtained from Eq. (A14) by using Eq. (25), which was proved earlier.

Equations (28)–(30) are obtained by substitution between Eqs. (25), (26), and (27).

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[10] The name derives from the fact that the differential $2\pi d\Psi$ represents the flux across the infinitesimal annulus obtained by rotating the line element between (r,θ) and $(r+dr,\theta+d\theta)$ about the symmetry axis. The surface Ψ=const is a magnetic surface. Similarly, in hydrodynamics the surface of revolution obtained by setting the stream function equal to a constant is composed of stream lines. See, for example, H. K. Moffatt, *Magnetic Field Generation in Electrically Conducting Fluids* (Cambridge University Press, Cambridge, England, 1978).

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