Flux-vortex structure in type-II superconductors carrying a longitudinal current

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For values of r greater than the coherence length ξ, the axially symmetric Ginzburg-Landau equations are solved for a flux vortex carrying a longitudinal current. The field is not force free, and it is shown that there are no regular solutions to the force-free field equations that decay exponentially with increasing penetration into a superconductor. It is also shown, in this approximation, that in the case of a vortex carrying a nonzero longitudinal current, the Ginzburg-Landau equations are equivalent to the radial pressure-balance equilibrium relation in ideal magnetohydrodynamics. The techniques developed in this field to address stability issues can then be used to answer questions related to vortex stability.

INTRODUCTION

The solution to the Ginzburg-Landau equations found by Abrikosov[1] has been used extensively to study the properties of the flux lattice of type-II superconductors when the transport current and an applied magnetic field are directed perpendicular to each other. When this solution is used to study the case where the current is parallel to the field, the behavior of the vortices is not well understood. The observation of a longitudinal paramagnetic moment suggests, however, that the current flows in a helical path. Various theoretical models have been discussed in the reviews by Campbell and Evetts[2] and Timms and Walmsley.[3]

Because longitudinal critical currents are found to be significantly larger than those in a transverse field,[4] Bergeron[5] has suggested that the current adopts a force-free configuration. For this to be true, the configuration of the flux-line lattice itself must be force free and the current must flow along the helical flux lines, parallel to the magnetic field of the Abrikosov vortices. Since the Lorentz force between the current and the field of the vortex would then vanish, it is often said that such a configuration is force free. It is shown here that, for values of $r>\xi$, the Ginzburg-Landau equations do not have a force-free solution for the vortex itself (the configuration of the flux line lattice is a separate question), and that there are no regular solutions to the force-free field equations that decay exponentially with increasing distance into a superconductor in contradistinction to the London theory.

The Ginzburg-Landau equations do, however, have a twisted field solution that is not force free and which could be used as a model for a vortex carrying a longitudinal current. For the De Gennes–Matricon[6] model of a vortex where there is a normal core of radius approximately equal to ξ, most of this current would flow in the region $\xi \le r \le \lambda$ where the De Gennes–Matricon model assumes the vortex of superconducting electrons flow around the normal state core, and the fields and currents obey the London equations.

It is also shown that for a vortex carrying a nonzero longitudinal current, the Ginzburg-Landau equations (for $r>\xi$) in cylindrical coordinates are equivalent to the condition for pressure equilibrium in ideal magnetohydrodynamics. The solution to these equations is known to exhibit a corkscrew instability due to the twisting of the magnetic field. Vortex instability was predicted by Clem,[7] who also noticed the analogy to magnetohydrodynamics. However, the vortex model used by him did not itself have a twisting field.[8]

THE GINZBURG-LANDAU EQUATIONS

The dimensionless form of the Ginzburg-Landau equations is[9]

$$-\nabla\times(\nabla\times\mathbf{A})=|\psi|^2\mathbf{A}+\tfrac{1}{2}i\kappa^{-1}(\psi^*\nabla\psi-\psi\nabla\psi^*)\;,$$
$$(i\kappa^{-1}\nabla+\mathbf{A})^2\psi=\psi(1-|\psi|^2)\;. \tag{1}$$

By letting the order parameter be $\psi=fe^{i\varphi}$, the first equation reduces to

$$-\nabla\times(\nabla\times\mathbf{A})=f^2(\mathbf{A}-\kappa^{-1}\nabla\varphi)\;. \tag{2}$$

Abrikosov introduces a scalar quantity Q which is the magnitude of the vector $\mathbf{A}-\kappa^{-1}\nabla\varphi$. Consider instead, the introduction of a *vector* quantity $\mathbf{Q}=\mathbf{A}-\kappa^{-1}\nabla\varphi$. Since $\nabla\times\mathbf{Q}=\nabla\times\mathbf{A}=\mathbf{B}$, Eq. (2) can be written as

$$-\nabla\times(\nabla\times\mathbf{Q})=f^2\mathbf{Q}\;. \tag{3}$$

In cylindrical coordinates, Eq. (3) results in two equations,

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(rQ_\phi)\right]=f^2Q_\phi\;,$$
$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dQ_z}{dr}\right]=f^2Q_z\;, \tag{4}$$

which are valid when the flux vortices are sufficiently separated to have axial symmetry.

Note that $\mathbf{Q}=\mathbf{A}-\kappa^{-1}\nabla\varphi$ is not a true gauge transformation since φ is a phase.[10] Consider the line integral of $\mathbf{Q}$ around a closed path Γ,

$$\oint_\Gamma \mathbf{Q}\cdot dl = \oint_\Gamma \mathbf{A}\cdot dl - \kappa^{-1}\oint_\Gamma \nabla\varphi\cdot dl \ . \tag{5}$$

Since φ is a multivalued function, it can be made single valued by introducing a cut defined by, for example, $\phi=0$. Now for any two points p_1 and p_2, the difference in the value of the scalar function φ is

$$\varphi_{p_2}-\varphi_{p_1}=\int_{p_1}^{p_2}\nabla\varphi\cdot dl \ .$$

If the path of integration is closed, but does not cross the cut needed to make ϕ single valued, the integral will vanish by Stokes' theorem.[11] Now as $r\to\infty$, $\mathbf{Q}\to 0$, so that if the contour Γ is chosen at infinity, and the path of integration Γ crosses the cut,

$$\oint_\Gamma \nabla\varphi\cdot dl = \kappa\oint_\Gamma \mathbf{A}\cdot dl \ .$$

The last integral is the total flux through Γ. Since φ is a phase, it must change by 2π each time Γ is traversed. Thus, the integral $\oint_\Gamma \mathbf{A}\cdot dl$ must be a multiple n of a fixed quantity of flux $\Phi_0=2\pi/\kappa$, so that $\oint_\Gamma \nabla\varphi\cdot dl = n\kappa\Phi_0$. It can be shown[12] that the value of this integral, while it depends on n, is not dependent on the particular path Γ. Thus, for an arbitrary path Γ, Eq. (5) can be written as $\oint_\Gamma \mathbf{Q}\cdot dl=\Phi-n\Phi_0$, where Φ is the magnetic flux through the surface bounded by the contour Γ. n is the winding number which can be defined, for example, when the path Γ is contained in the plane $z=0$ as

$$n=\frac{1}{2\pi}\int_0^{2\pi}\frac{d\varphi(\phi)}{d\phi}d\phi \ .$$

The winding number n is also known as the topological charge.

Following Abrikosov, if φ is set equal to the azimuthal coordinate ϕ, the second of Eqs. (1) becomes

$$-\kappa^{-2}\frac{1}{r}\frac{d}{dr}\left[r\frac{df}{dr}\right]+Q^2 f=f(1-f^2) \ , \tag{6}$$

where, despite the formal similarity, this differs from the usual result since here $Q^2=Q_\phi^2+Q_z^2$. In what follows, attention will be restricted to values of r sufficiently far from the vortex core ($r>\kappa^{-1}$ in Ginzburg-Landau units) so that $f\sim 1$, and Eq. (6) will play no further role.

With $f=1$ and the boundary condition that $\mathbf{Q}\to 0$ as $r\to\infty$, Eqs. (4) have the solutions

$$\begin{aligned} Q_\phi &= K_1(r) \ , \\ Q_z &= cK_0(r) \ , \end{aligned} \tag{7}$$

where the K_n are modified Bessel functions of the second kind, c is a constant, and the constant associated with Q_ϕ has been set equal to unity so that the solution will reduce to Abrikosov's for $c=0$. The magnetic field and current associated with $\mathbf{Q}=\{0,K_1(r),cK_0(r)\}$ are readily calculated to be[13]

$$\begin{aligned} \mathbf{B} &= \{0, cK_1(r), -K_0(r)\} \ , \\ \mathbf{J} &= \{0, -K_1(r), -cK_0(r)\} \ . \end{aligned} \tag{8}$$

It is easily verified that Eqs. (8) *do not* satisfy, for real values of α, the force-free relation[14]

$$\nabla\times\mathbf{B}=\alpha\mathbf{B} \ , \tag{9}$$

where α is, in general, a scalar function of position. The force-free relation given by Eq. (9) is a direct consequence of the requirement that, within some region, the magnetic field be everywhere parallel to the direction of the current flow; i.e., satisfy the condition that $(\nabla\times\mathbf{H})\times\mathbf{H}=0$.

Note that the cosine of the angle between $\mathbf{J}$ and $\mathbf{B}$ given by Eqs. (8) is

$$c\{[K_0(r)]^2-[K_1(r)]^2\}/\{[K_1(r)]^2+c^2[K_0(r)]^2\}^{1/2} \times\{[K_0(r)]^2+c^2[K_1(r)]^2\}^{1/2} \ ,$$

so that using the relation

$$\lim_{r\to 0} K_n(r)\cong\tfrac{1}{2}\Gamma(n)\left[\frac{r}{2}\right]^{-n}$$

it can be seen that $\mathbf{J}$ and $\mathbf{B}$ are perpendicular for values of r large compared to unity and become parallel or antiparallel (depending on the sign of c) for small values of r. The solution is not valid, of course, for values of $r\le\xi$. The variation in currents and fields over this region, where Eqs. (4) and (6) must apply with $f=f(r)$, must be such as to match the axial field at $r=0$ and the solution given by Eqs. (8) for $r>\xi$.

FORCE-FREE FIELDS

For axially symmetric fields such that $\mathbf{B}=\{0,B_\phi(r),B_z(r)\}$, the force-free condition of Eq. (9) can be written in the form

$$\frac{d}{dr}(B_\phi^2+B_z^2)+2\frac{B_\phi^2}{r}=0 \tag{10}$$

with α being given by

$$\alpha(r)=\frac{1}{B_z}\frac{1}{r}\frac{d(rB_\phi)}{dr} \ . \tag{11}$$

Any function $g(r)=B_\phi^2+B_z^2$ such that $g(r)\ge 0$, $g'(r)\le 0$, and $d[r^2 g(r)]/dr\ge 0$ will give a nonsingular solution of Eq. (10). For distances r greater than the coherence length, the London equations are expected to govern the fields and currents and the field would have to decay exponentially if a force-free model of a vortex was possible to construct.

If one attempts to use the relations given by Eqs. (10) and (11) to find a force-free configuration for a flux vortex, with the condition that the magnitude of the field decay exponentially with the distance r into the superconductor, the last inequality given above is violated for some value of r. For example, let $g(r)=B_0 e^{-r/\lambda}$. Equation (10) is easily solved to give

$$\begin{aligned} B_\phi &= B_0 e^{-r/\lambda}\left[\frac{r}{\lambda}\right]^{1/2} \ , \\ B_z &= B_0 e^{-r/\lambda}\left[1-\frac{r}{\lambda}\right]^{1/2} \ . \end{aligned} \tag{12}$$

B_z is imaginary for $r>\lambda$, and $\alpha(r)$ is singular at $r=\lambda$. For $r>\lambda$,

$$\frac{d[r^2g(r)]}{dr}=1-\frac{r}{\lambda}<0\ .$$

Thus a nonsingular force-free model for a vortex, where the magnitude of the field decays exponentially for distances greater than the coherence length, is not possible.

THE GINZBURG-LANDAU EQUATIONS AND IDEAL MAGNETOHYDRODYNAMICS

The fact that the Ginzburg-Landau equations are equivalent to ideal-fluid hydrodynamics and London electrodynamics was shown by Fröhlich.[15] This can be demonstrated here by rewriting Eqs. (4) with $f=1$ as

$$Q_\phi=\frac{dB_z}{dr},\quad Q_z=-\frac{1}{r}\frac{d(rB_\phi)}{dr}\ . \tag{13}$$

Multiplying the first of these equations by H_z, the second by H_ϕ, and subtracting the second from the first,

$$\frac{d}{dr}\left[\frac{B_\phi^2+B_z^2}{2}\right]+\frac{B_\phi^2}{r}=B_zQ_\phi-B_\phi Q_z\ . \tag{14}$$

The right-hand side of this equation is just the $\hat{\mathbf{r}}$ component of $\mathbf{Q}\times\mathbf{B}$, the only component that does not vanish. From the vector relation

$$(\nabla\times\mathbf{H})\times\mathbf{H}=(\mathbf{H}\cdot\nabla)\mathbf{H}-\nabla\left[\frac{H^2}{2}\right],$$

it is readily seen that $\mathbf{Q}\times\mathbf{B}=-\mathbf{J}\times\mathbf{B}$. If $\mathbf{B}\times\mathbf{Q}$ is viewed as the gradient of a pressure P, Eq. (14) can be written as

$$\frac{d}{dr}\left[P+\frac{B_\phi^2+B_z^2}{2}\right]+\frac{B_\phi^2}{r}=0\ . \tag{15}$$

For the solutions for $\mathbf{Q}$ and $\mathbf{B}$ given by Eqs. (7) and (8) the $\hat{\mathbf{r}}$ component of $\mathbf{Q}\times\mathbf{B}$ may be calculated to be

$$\hat{\mathbf{r}}\cdot(\mathbf{Q}\times\mathbf{B})=-(1+c^2)K_1(r)K_0(r)=\frac{(1+c^2)}{2}\frac{dB_z^2}{dr}\ . \tag{16}$$

P is then

$$P=-\frac{(1+c^2)}{2}B_z^2\ . \tag{17}$$

Note that ∇P is in the positive $\hat{\mathbf{r}}$ direction. Equation (15) is the radial pressure-balance equilibrium relation in ideal magneto-hydrodynamics.[16] For $c=0$, the latter is satisfied identically.

There are some general constraints implied by the form given by Eq. (14) for Eqs. (4). If one defines

$$g(r)=P+\frac{B_\phi^2+B_z^2}{2}\ , \tag{18}$$

Eq. (15) can be written as

$$B_\phi^2=-r\frac{d}{dr}g(r)\ . \tag{19}$$

From Eqs. (18) and (19),

$$B_z^2=2[g(r)-P]+r\frac{d}{dr}g(r)\ . \tag{20}$$

Thus, B_ϕ and B_z will be real if $(d/dr)g(r)\le 0$ and if

$$\frac{d[r^2g(r)]}{dr}\ge 2rP\ .$$

While the solution given by Eqs. (8) and (17) satisfy both these inequalities as well as $g(r)\ge 0$, it is also true that

$$\frac{d[r^2g(r)]}{dr}\le 0\ .$$

However, unlike the case of a force-free field, the solution given by Eqs. (8) is regular for all values of r.

The equivalency of the Ginzburg-Landau equations for a vortex carrying a nonzero longitudinal current and the radial pressure-balance equilibrium relation in ideal magnetohydrodynamics means that the full panoply of techniques developed in the latter field are available to address the question of vortex stability. And, while in general this is a difficult problem, it is clear that a vortex will be subject to a kink instability for very small amounts of twisting. The vortex can be expected to become unstable[17] at a value of axial current such that

$$\frac{\langle B_\phi\rangle}{\langle B_z\rangle}\approx 2\pi\frac{\lambda}{L}\ ,$$

where λ is the radius of a vortex (approximately the penetration depth) and L the length of the vortex.

SUMMARY

By introducing a vector quantity $\mathbf{Q}=\mathbf{A}-\kappa^{-1}\nabla\varphi$, it is possible to solve the Ginzburg-Landau equations for a twisted field that is not force free [Eqs. (8)] and which could be useful as a model of a vortex carrying a longitudinal current. When this current vanishes, the solution reduces to that of Abrikosov. Because of the equivalency of the Ginzburg-Landau equations for a vortex carrying a nonzero longitudinal current with the radial pressure-balance equilibrium relation in ideal magnetohydrodynamics, it is clear that the vortex will become unstable with respect to a kink instability for very small amounts of twisting.

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