FORCE-FREE MAGNETIC FIELDS

SOLUTIONS, TOPOLOGY AND APPLICATIONS

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World Scientific
Singapore • New Jersey • London • Hong Kong

Published by

World Scientific Publishing Co. Pte. Ltd.

P O Box 128, Farrer Road, Singapore 912805

USA office: Suite 1B, 1060 Main Street, River Edge, NJ 07661

UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library.

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ISBN 981-02-2497-4

This book is printed on acid-free paper.

Printed in Singapore by Uto-Print

PREFACE

The study of force-free magnetic fields has its origin in astrophysics. In a plasma where the flow velocity is much smaller than both the isothermal sound and Alfvén velocities, the equation of motion of the plasma reduces to that of magnetohydrostatics. When, in addition, both gravity and hydrodynamic forces are negligible compared to local magnetic stresses, the force-balance relation of magnetohydrostatics becomes the force-free field equation.

This book begins with an introductory chapter that outlines the history of force-free magnetic fields, and the relation of such fields to hydrodynamics and astrophysics. While force-free configurations can eliminate magnetic stresses in a given region, these stresses cannot be canceled everywhere. For finite force-free magnetic field configurations the virial theorem, which can be used to relate the stored magnetic energy to the integral of the trace of the stress tensor over the magnetic field volume, sets limits that must be satisfied. This is discussed in Chapter 2. Various techniques of obtaining solutions to the force-free field equations, and the solutions themselves, are the subject of Chapter 3.

The solutions to the force-free field equations discussed in Chapter 3 have the common feature of being "twisted" fields where the field lines are twisted with respect to each other, and may be knotted. This mandates a discussion of field line topology and the concept of helicity. For highly symmetric field configurations, the field lines may lie on magnetic surfaces. Cowling's theorem asserts that such surfaces cannot be simply connected. Magnetic field topology and the concept of helicity are dealt with in Chapter 4, while the computation of magnetic energy in multiply connected domains is the subject of Chapter 5. These chapters introduce and use some fundamental concepts from both knot theory and algebraic topology. Although the book assumes some knowledge of differential forms, an attempt has been made to abstract only those elements of knot theory and algebraic topology that are explicitly needed for explication of the physics. While the principal focus of the book is force-free magnetic fields, the topics of field topology, helicity, and magnetic energy in multiply connected domains have application to a far wider spectrum of physical problems.

The discussion of force-free fields in relation to superconductivity makes its appearance because of the claim in the relevant literature that force-free fields play an important role in type-II superconductors. When a transport current is applied to a type-II superconductor with an applied magnetic field parallel to the current, it is often maintained that the flux lattice, formed when the field strength exceeds the lower critical field, assumes a force-free configuration. Experimental observations suggest that the current must be carried along the vortices comprising the flux-line lattice. If this is true, the Ginzburg-Landau equations should allow a solution for a vortex carrying a longitudinal current. That this is indeed the case is shown in Chapter 6, which also discusses some applications of force-free fields to solar prominence models.

In past years, there has also been great interest in force-reduced magnets for both thermonuclear and energy storage applications where high magnetic fields are needed. More recently, there has been renewed interest in force-reduced field configurations due to the discovery of high-temperature superconductors. Because the materials that exhibit superconductivity at high temperature are ceramics, and tend to be quite brittle, it is important to reduce the forces they might experience if they are to be used for high-field applications. Force-free configurations also have the potential to raise the critical magnetic field and current-density limits in superconductors.

I would like to express my thanks to Professor Charles Nissim-Sabat for reading the penultimate version of the manuscript and making several valuable suggestions and corrections.

CONTENTS

Preface v

Chapter 1. INTRODUCTION 1
- 1.1. Force-Free Magnetic Fields and Hydrodynamics 4
- 1.2. Relevance of Force-Free Magnetic Fields to Astrophysics and Solar Physics 6

Chapter 2. THE VIRIAL THEOREM 9
- 2.1 The Virial Theorem Applied to Magnetic Fields 10
  - 2.1.1. Virial Condition in Terms of the Magnetic Field Vectors 11
  - 2.1.2. Virial Condition in Terms of the Maxwell Stress Tensor 13
- 2.2 Practical Applications of the Virial Theorem 14

Chapter 3. SOLUTIONS TO THE FORCE-FREE FIELD EQUATIONS 17
- 3.1 Generation of Solutions to the Force-Free Field Equations from Solutions to the Wave Equation 21
  - 3.1.1 The "Vector" Wave Equation 21
  - 3.1.2 The "Scalar" Wave Equation 22
  - 3.1.3 Wave Equation Solutions for Ballabh's Generalization of the Force-Free Field Equations 24
- 3.2 Non-Constant α Solutions to the Force-Free Field Equations in Spherical and Cylindrical Coordinates 25
  - 3.2.1 Flux Functions, Toroidal and Poloidal Fields 26
  - 3.2.2 Spherically Symmetric Solutions 27
    - 3.2.2.1 Separation of the Grad-Shafranov Equation in Terms of an Arbitrary Function 28
    - 3.2.2.2 Solution of the Separated Grad-Shafranov Equation 29

3.2.3 Axially Symmetric Solutions 33
3.2.3.1 The Grad-Shafranov Equation for Axial Symmetry 34
3.2.3.2 A Hierarchical Set of Solutions for a Special Class of Fields 35
3.2.3.2.1 Solution for the Case where $\boldsymbol{H} = [0, H_\phi(r), H_z(r)]$ 36
3.2.3.2.2 An Hierarchical Set of Solutions 38
3.3 Solutions in Toroidal Coordinates 41
3.3.1 Toroidal Coordinates 43
3.3.2 Non-Constant α Field Equations 45
3.3.3 Solution in Toroidal Coordinates for Constant α 46

Chapter 4. FIELD TOPOLOGY 57
4.1 Magnetic Surfaces 59
4.1.1. Topology of Non-Constant α Solutions 60
4.1.2. Topology of Constant α Solutions 62
4.2 Magnetic Field Helicity 64
4.2.1. Twist, Kink and Link Helicity 65
4.2.2. Helicity and Locality 68
4.2.3. The Classical and Asymptotic Hopf Invariant 70
4.2.3.1 The Classical Hopf Invariant 72
4.2.3.2 The Asymptotic Hopf Invariant 75

Chapter 5. MAGNETIC ENERGY IN MULTIPLY CONNECTED DOMAINS 83
5.1 Gauge Invariance 88
5.2 De Rham's Theorems and Homology 89
5.3 Physical Interpretation of the Magnetic Energy Boundary Term 93
5.3.1. Physical Examples 96
5.3.1.1. Ring Current in a Toroidal Domain 96
5.3.1.2. Application to Force-Free Fields 98

Chapter 6. APPLICATIONS 103
6.1 Solar Prominence Models 104
6.2 Type II Superconductors 108

Appendix 1. FORCE-FREE FIELDS AND ELECTROMAGNETIC WAVES 119
A1.1 Electromagnetic Fields with $\boldsymbol{E}$ Parallel to $\boldsymbol{B}$ 119
A1.2 The Propagation of Electromagnetic Waves in Optically Active Media 124

Appendix 2. PROOF OF THE JACOBI POLYNOMIAL IDENTITIES 129

Appendix 3. SEPARATION OF THE WAVE EQUATION, CYCLIDES, AND BOUNDARY CONDITIONS 133
A3.1 Separation of the Wave Equation 133
A3.2 Cyclides 143
A3.3 Boundary Conditions for the Helmholtz and Vector Wave Equation 151

INDEX 155

Chapter 1

INTRODUCTION

Force-free magnetic fields satisfy the condition that the magnetic field in some region is everywhere parallel to the direction of current flow. Such fields are force-free in the sense that the Lorentz force density vanishes. This condition can be written as

$$(\nabla \times \boldsymbol{H}) \times \boldsymbol{H} = 0, \tag{1.1}$$

where it is assumed throughout this book, except for parts of Appendix 1 where the relation of the force-free condition to the propagation of electromagnetic waves in optically active media is discussed, that $\boldsymbol{B} = \mu_0 \boldsymbol{H}$. So as to maintain consistency with the literature, and for convenience, both $\boldsymbol{B}$ and $\boldsymbol{H}$ will be used throughout the book. For Eq. (1.1) to hold, $\boldsymbol{H}$ must satisfy

$$\nabla \times \boldsymbol{H} = \alpha \boldsymbol{H}, \tag{1.2}$$

where α may be a function of position. Note that generally force-free fields do not obey the superposition principle, where the sum of any two solutions to the field equations is itself a solution. That is, if $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ satisfy the force-free condition so that $\nabla \times \boldsymbol{H}_1 = \alpha_1 \boldsymbol{H}_1$ and $\nabla \times \boldsymbol{H}_2 = \alpha_2 \boldsymbol{H}_2$, then $\nabla \times (\boldsymbol{H}_1 + \boldsymbol{H}_2) = \alpha_1 \boldsymbol{H}_1 + \alpha_2 \boldsymbol{H}_2$. The superposition principle will hold only if the additional condition $\alpha_1 = \alpha_2$ is imposed.

Since $\boldsymbol{H}$ is a magnetic field, it must satisfy the divergence condition

$$\nabla \cdot \boldsymbol{H} = 0, \tag{1.3}$$

and it therefore follows from Eq. (1.2) that

$$\nabla \alpha \cdot \boldsymbol{H} = 0. \tag{1.4}$$

This means that α must be a constant on any given field line, and if the field lines cover a surface, then α is a constant on that surface. Physically, since Eq. (1.2) can be written as $\boldsymbol{J} = \alpha \boldsymbol{H}$, taking the divergence of both sides of this relation shows that Eq. (1.4) is simply a statement of the conservation of electric current. Additional discussion of Eq. (1.4) appears in the introductory paragraphs to Chapter 4. It should, however, be noted here that in general the equation $\nabla \alpha \cdot \boldsymbol{H} = 0$ has two functionally independent solutions, in terms of which one may write any other solution. The level sets of any such solution constitute the magnetic surfaces.

The concept of magnetic helicity is fundamental to any discussion of the energy contained in force-free magnetic fields. It has been proved by Woltjer,[1] and under somewhat less restrictive conditions by Moffatt,[2] that the equations of non-dissipative magnetohydrodynamics have the magnetic helicity,

$$\mathcal{H} = \int_V \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V \tag{1.5}$$

as a constant of the motion. The appearance of the vector potential $\boldsymbol{A}$ in the definition of helicity immediately raises the question of gauge invariance, as well as how the gauge term should be interpreted in multiply connected domains. It will be shown in Chapter 4 that the helicity introduced by the gauge term explicitly depends on the topology and has an important physical interpretation.

Magnetic helicity has been interpreted as a measure of the degree of linkage of the field lines, but this definition cannot be taken too literally. Helicity can be non-localizable in the same sense that the magnetic energy of a magnetostatic field is not localizable. Helicity has also been identified with the asymptotic Hopf invariant,[3] and the topological interpretation of helicity in terms of the Gauss linking number and its limiting form, the Călugăreanu invariant, has been extensively discussed by Moffatt and Ricca.[4] Other integrals of the equations of motion of non-dissipative magnetohydrodynamics have been discussed by Woltjer[5] and Kruskal and Kulsrud.[6] These issues are extensively discussed in Chapter 4.

The question of the magnetic energy contained in force-free fields was initially addressed by Chandrasekhar and Woltjer.[7] In minimizing the total magnetic energy with the constraint that helicity be conserved, Woltjer introduced a constant, α, through the method of Lagrangian multipliers, to show that the variational problem

$$\delta \int_V \left\{ |\nabla \times \boldsymbol{A}|^2 - \alpha \, \boldsymbol{A} \cdot (\nabla \times \boldsymbol{A}) \right\} \mathrm{d}V = 0 \tag{1.6}$$

implies that the force-free magnetic field equations

$$\nabla \times \boldsymbol{B} = \alpha \boldsymbol{B} \tag{1.7}$$

are satisfied. In this way, Woltjer succeeded in showing that force-free fields with constant α correspond to the lowest state of magnetic field energy which a closed system may attain.[8] Freedman[9] has addressed the issue of a lower bound to the energy. The computation of magnetic energy in finite, multiply connected domains such as a torus involves some subtleties that are discussed in Chapter 5.

Taylor,[10] in discussing the relaxation of a toroidal plasma, pointed out that for the case where the field generates magnetic surfaces, and where the conductivity is assumed to be infinite, there is one invariant of the form given by Eq. (1.5) for each surface. He then noted that the magnetic field that results from minimizing the energy, for all variations $\delta \boldsymbol{A}$ which leave this set of invariants unchanged, satisfies

$$\nabla \times \boldsymbol{B} = \alpha(\boldsymbol{r}) \boldsymbol{B} \tag{1.8}$$

where $\alpha(\boldsymbol{r})$ is now a function of position, but is constant on each of the magnetic surfaces.

Woltjer's assumption of ideal magnetohydrodynamics means that the constant α state is topologically inaccessible from most initial configurations. To make the minimum energy state accessible, Taylor conjectured that global helicity could be conserved while finite diffusivity effects are allowed to relax the toroidal plasma to a state of minimum energy. The magnetic energy and topological invariants other than global helicity are assumed to decay on a shorter time scale than that associated with the decay of global helicity. While the Taylor hypothesis is supported by reversed-field pinch confinement experiments,[11] it raises a fundamental theoretical problem: If the helicity invariant for each magnetic surface is allowed to change so that relaxation is possible, how can global helicity be conserved? This question has received much attention,[12] but has not yet been completely resolved.[13]

For an ideal toroidal plasma, Bhattacharjee, *et al.*[14] have established the existence of an infinity of invariants of the form

$$K_\alpha = \frac{1}{2} \int_V \chi^\alpha \boldsymbol{A} \cdot \boldsymbol{B} \, \mathrm{d}V \tag{1.9}$$

for a particular class of ideal variations. These are defined as

$$\begin{aligned} \delta\Psi &= -\xi \cdot \nabla \Psi \\ \delta\Phi &= q(\Psi) \delta\Psi, \end{aligned} \tag{1.10}$$

where Ψ and Φ are, respectively, the poloidal and toroidal flux functions, and ξ is a virtual displacement. Φ is the flux through a cross section of the magnetic surface and Ψ is the flux through a ribbon, of which one edge is the magnetic axis (the magnetic surface having zero volume where it is assumed that $\Phi = 0$) and the other lies on the magnetic surface (going around the torus the long way). [See also Section 3.2.1.] In Eq. (1.9), χ is the

helical flux function given by $\chi = q_s\Psi - \Phi$ and q_s is a rational number of the form m/n that determines the helical pitch. The argument is now made that because the global helicity K_0 is the only member of the infinite set K_α that is independent of the pitch of the assumed helical symmetry, only K_0 will be conserved when disturbances of all helical pitch lengths are present.

The issue of whether the topological invariants other than helicity can be destroyed during the relaxation process has also been addressed by Freedman and Berger.[15] They approach the problem by means of a theorem in the theory of weighted, framed links that gives the number of flux reconnections needed to reach a standard form. The framed links considered are defined to have a rational framing or self linking number. Their findings are generally consistent with Taylor's theory of relaxation, although they do suggest that the stability of positively braided flux tubes with positive subhelicities could prevent the field from reaching the minimum energy state.

1.1. Force-Free Magnetic Fields and Hydrodynamics

The foundation for much of the work on force-free magnetic fields can be found in the hydrodynamical literature. There, the analog to the force-free magnetic field condition appears as $\nabla \times \boldsymbol{v} = \Omega\, \boldsymbol{v}$, where $\boldsymbol{v}$ is the fluid velocity. Solutions to this equation, where Ω is a function of position, are known as Beltrami fields; solutions for the case where Ω is a constant are known as Trkalian fields. Bjorgum[16,17] has written two comprehensive papers on these types of fluid flow and has also given a method of finding Trkalian fields from solutions to the scalar wave equation. This is essentially the same method later developed by Chandrasekhar and Kendall[18] for application to force-free magnetic fields. A detailed exposition of this method, along with a generalization to the type of fields considered by Ballabh,[19] is given in Chapter 3.

If $\boldsymbol{F}$ is an external force per unit mass, the equation of motion of a homogeneous and incompressible inviscid fluid is

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \boldsymbol{F} - \frac{1}{\rho}\nabla p, \tag{1.11}$$

where the left side of this equation is the Stokes or convective derivative of $\boldsymbol{v}$, and p, $\boldsymbol{v}$, and ρ are respectively the pressure, velocity, and density. Using the vector relation $(1/2)\nabla \boldsymbol{v}^2 = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \boldsymbol{v} \times (\nabla \times \boldsymbol{v})$, and assuming that $\boldsymbol{F}$ is conservative so that it is derivable from a potential Π, Eq. (1.11) may be written as

$$\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = -\nabla\Pi - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla \boldsymbol{v}^2. \tag{1.12}$$

By dotting both sides of this equation with $d\boldsymbol{r}$, and noting that $d\boldsymbol{r}\cdot\nabla = d$, the second term on the right-hand side becomes $\frac{dp}{\rho}$, which may be written as $d\boldsymbol{r} \cdot \nabla \int \frac{dp}{\rho}$.

Thus, since $d\boldsymbol{r}$ is arbitrary, the right-hand side of Eq. (1.12) may be expressed as the gradient of a potential χ,

$$\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = -\nabla\chi, \tag{1.13}$$

where

$$\chi = \int \frac{dp}{\rho} + \Pi + \frac{\boldsymbol{v}^2}{2}. \tag{1.14}$$

For the case of steady flow, $\partial \boldsymbol{v}/\partial t = 0$, and by taking the curl of both sides of Eq. (1.13) one obtains the vorticity equation, or condition of integrability,

$$\nabla \times \left[\boldsymbol{v} \times (\nabla \times \boldsymbol{v})\right] = 0. \tag{1.15}$$

A similar result may be obtained for the steady flow of a compressible inviscid medium.[20,21] Equation (1.15) is satisfied identically if $\boldsymbol{v}$ is a Beltrami or Trkalian field.

For incompressible flow, the equation of continuity implies that $\nabla\cdot\boldsymbol{v} = 0$. Thus, there is an exact analogy between the velocity $\boldsymbol{v}$ and the magnetic field intensity $\boldsymbol{H}$, and the vorticity $\nabla \times \boldsymbol{v}$ and the current density $\boldsymbol{J}$. The analogy is exact in the sense that solutions to $\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = 0$ correspond to solutions to Eq. (1.1), provided the boundary conditions are compatible. An example is the analogy between Hill's spherical vortex[22] and the magnetohydrodynamical "spheromak."[23]

Beltrami fields are of interest in hydrodynamics because they serve as the link between irrotational flow where $\nabla \times \boldsymbol{v} = 0$ and the more general rotational flow where $\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) \neq 0$. In aerodynamics, Lanchester and Prandtl[24] have shown that the tip (or trailing) vortices at the end of a wing satisfy the force-free relation. This is a direct consequence of Stokes' circulation theorem, which implies that the circulation about the tip vortices must be equal to the circulation about the airfoil to which they are attached. Steady motion is only possible when the vortex lines coincide with the lines of flow; i.e., when $\boldsymbol{v} \times (\nabla \times \boldsymbol{v}) = 0$. If the circulation is not constant along the span of the airfoil, there is a trailing vortex sheet that can be thought of as rolling up into two tip vortices with circulation equal to the circulation around the median section of the airfoil.

In the context of inviscid and incompressible fluids, the concept of helicity has been generalized to higher dimensional manifolds. At least one generalized helicity integral has been shown to exist if the manifold is odd-dimensional, and an infinite number of integral invariants exist for an arbitrary even-dimensional manifold. By taking the curl of Eq. (1.11) so as to obtain the vorticity equation, Olver[25] showed that the Euler equation has a Hamiltonian structure, and that the degeneracies of the Hamiltonian operator lead to the conservation of helicity. This result was generalized to $\mathbb{R}^n$ by Serre,[26] and later by Khesin and Chekanov[27] to arbitrary odd-dimensional manifolds. In the general case, the Euler equation has the integral invariant

$$I = \int_{M^n} v \wedge (dv)^m \tag{1.16}$$

where $n = 2m + 1$. For $m = 1$, the manifold is 3-dimensional and I is the helicity. Here, the velocity is written in terms of the 1-form v so that dv is the vorticity and $v \wedge dv$ corresponds to the inner product of the velocity and vorticity.

1.2. Relevance of Force-Free Magnetic Fields to Astrophysics and Solar Physics

The possibility that force-free magnetic fields might exist in an astrophysical context was apparently first considered by Lüst and Schlüter.[28] In plasma physics, the condition for pressure equilibrium is given by

$$\boldsymbol{H} \times (\nabla \times \boldsymbol{H}) = \nabla p. \tag{1.17}$$

For many astrophysical applications, the gas pressure is small enough to allow the right hand side of Eq. (1.17) to be set equal to zero. Nevertheless, large currents may still be able to flow by virtue of the high electrical conductivity of low density ionized gases. For example, the magnetic field associated with a solar prominence is often modeled as a twisted flux tube. This is due to the fact that, while material motions dominate in the sun's photosphere or below, in the chromosphere, and above in the corona, the gas pressure is negligible, and the field must be approximately force-free.

Twisting of the flux tube can be introduced either by photospheric twisting motions (due to Coriolis forces) at the locations where the base of the arch enters the photosphere,[29] or by flux cancellation; i.e., by shear and convergence along the neutral line separating two regions of opposite magnetic polarity, followed by reconnection of the field lines.[30] In either case, currents are introduced that flow along the field lines.

When a plasma has finite conductivity, constant α force-free configurations have the unique[31] property of allowing resistive decay of the field without introducing magnetic stresses that act on the plasma.[32] For this to be true, α must be constant in both space and time. That a decaying force-free field remains force free is readily seen by writing the force-free relation as $\boldsymbol{J} = \alpha \boldsymbol{H}$ and taking the curl to obtain $\nabla \times \boldsymbol{J} = \alpha \boldsymbol{J}$. Now if $\boldsymbol{J} = \sigma \boldsymbol{E}$, the last relation becomes $\nabla \times \boldsymbol{E} = \alpha \boldsymbol{E}$. But $\alpha \boldsymbol{E} = (\alpha/\sigma)\boldsymbol{J} = (\alpha^2/\sigma)\boldsymbol{H}$ and $\nabla \times \boldsymbol{E} = -\mu_0 \dot{\boldsymbol{H}}$. This results in the partial differential equation

$$-\mu_0 \dot{\boldsymbol{H}} = \frac{\alpha^2}{\sigma} \boldsymbol{H}. \tag{1.18}$$

Assuming that the solution may be separated by writing $\boldsymbol{H} = \boldsymbol{H}_0(\boldsymbol{r}) f(t)$, where $\boldsymbol{H}_0$ is any time independent solution of the force-free field equation, results in the ordinary differential equation

$$\frac{f'}{f} = -\frac{\alpha^2}{\sigma \mu_0}. \tag{1.19}$$

The solution is therefore

$$\boldsymbol{H} = \boldsymbol{H}_0 \exp\left(-\frac{\alpha^2}{\sigma \mu_0} t\right). \tag{1.20}$$

The case where α is not constant has been investigated by Low.[33]

It should be clear from the above that force-free magnetic fields can be expected to play a significant role in astrophysics wherever magnetic fields not derivable from a potential (i.e., where the current density does not vanish) are present and the gas density is low; for example, in stellar envelopes, gaseous nebulae, and in interstellar space.

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Chapter 2

THE VIRIAL THEOREM

For finite force-free magnetic field configurations, the virial theorem sets limits on the volume that can be force-free. *While stresses may be eliminated in a given region, they cannot be canceled everywhere.* In general, force-free field configurations allow the forces needed to balance the outward pressure of the magnetic field to be reduced in magnitude by spreading them out over the bounding surface.

The virial theorem originally evolved out of the kinetic theory of gases, and is usually written in terms of a collection of mass points with position vectors $\boldsymbol{r}_i$ subject to applied forces $\boldsymbol{f}_i$ which include any forces of constraint. The Newtonian equations of motion are

$$\dot{\boldsymbol{p}}_i = \boldsymbol{f}_i. \tag{2.1}$$

Define $G = \sum_i \boldsymbol{p}_i \cdot \boldsymbol{r}_i$. Then,

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \sum_i (\dot{\boldsymbol{p}}_i \cdot \boldsymbol{r}_i) + (\boldsymbol{p}_i \cdot \dot{\boldsymbol{r}}_i) = \sum_i \boldsymbol{f}_i \cdot \boldsymbol{r}_i + 2T, \tag{2.2}$$

where T is the total kinetic energy. The term $\sum_i \boldsymbol{f}_i \cdot \boldsymbol{r}_i$ is known as the Virial of Claussius.[1] To obtain the usual statement of the virial theorem, one averages over the time interval τ,

$$\frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}G}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{\tau}\int_0^\tau \sum_i \boldsymbol{f}_i \cdot \boldsymbol{r}_i\,\mathrm{d}t + \frac{2}{\tau}\int_0^\tau T\,\mathrm{d}t, \tag{2.3}$$

or,

$$\frac{1}{\tau}\left[G(\tau) - G(0)\right] = \left\langle \sum_i \boldsymbol{f}_i \cdot \boldsymbol{r}_i \right\rangle + \langle 2T \rangle. \tag{2.4}$$

For periodic motion, the left hand side will vanish if τ is chosen to be the period. If G varies between finite limits, the left hand side will also vanish in the limit $\tau \to \infty$.

The virial theorem can be extended to a continuum density field of matter.[2] Instead of Eq. (2.1), one begins with $\dot{\boldsymbol{p}}(\boldsymbol{r}) = \boldsymbol{f}(\boldsymbol{r})$, where $\boldsymbol{p} = \rho \boldsymbol{v}$ is the local momentum density. Define G in this case to be

$$G = \int_V \boldsymbol{p} \cdot \boldsymbol{r} \, \mathrm{d}V . \tag{2.5}$$

Then the time average of $\frac{\mathrm{d}G}{\mathrm{d}t}$ is,

$$\frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}G}{\mathrm{d}t}\,\mathrm{d}t = \frac{1}{\tau}\int_0^\tau \left\{\int_V \boldsymbol{f}\cdot\boldsymbol{r}\,\mathrm{d}V\right\}\mathrm{d}t + \frac{2}{\tau}\int_0^\tau \left\{\int_V \rho v^2\,\mathrm{d}V\right\}\mathrm{d}t . \tag{2.6}$$

Now consider the limit as $\tau \to \infty$. Assuming that G varies between finite limits, the left hand side will again vanish leaving

$$\left\langle \int_V \boldsymbol{f}\cdot\boldsymbol{r}\,\mathrm{d}V \right\rangle + 2\langle T \rangle = 0. \tag{2.7}$$

2.1 The Virial Theorem Applied to Magnetic Fields

The virial theorem was first extended to include magnetic fields by Chandrasekhar and Fermi when addressing the question of the gravitational stability of infinitely conductive masses of cosmic dimensions in the presence of a magnetic field. They concluded that when the magnetic energy exceeds the gravitational potential energy, the configuration will become dynamically unstable.[3] Application of Eq. (2.7) is restricted here to equilibrium configurations where the kinetic energy vanishes. Since the forces in the virial term of Eq. (2.7) are then constant in time, the constraint imposed by the virial theorem is

$$\int_V \boldsymbol{f}\cdot\boldsymbol{r}\,\mathrm{d}V = 0. \tag{2.8}$$

Here $\boldsymbol{f}$ is the total force per unit volume exerted on the fluid. It consists of forces due to the magnetic field as well as the forces of constraint. Let the latter be designated by $\boldsymbol{f}$.

The next section discusses the implications of Eq. (2.8) in terms of the magnetic field vectors, while the following section uses the Maxwell stress tensor to obtain the same results. The first approach is due to Chandrasekhar[4] while the second follows Parker.[5]

2.1.1. Virial Condition in Terms of the Magnetic Field Vectors

Equation (2.8) can be written in terms of the magnetic field vectors as

$$\int_V \boldsymbol{f}\cdot\boldsymbol{r}\,\mathrm{d}V + \int_V [(\nabla\times\boldsymbol{H})\times\boldsymbol{B}]\cdot\boldsymbol{r}\,\mathrm{d}V = 0, \tag{2.9}$$

where $(\nabla\times\boldsymbol{H})\times\boldsymbol{B}$ is the Lorentz force density. Note that here the field is not assumed to be force-free. Consider the second integral in Eq. (2.9). Using the vector relation $(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c} = \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ this integral can be written as

$$\int_V (\nabla\times\boldsymbol{H})\cdot(\boldsymbol{B}\times\boldsymbol{r})\,\mathrm{d}V . \tag{2.10}$$

Now, integrating the vector relation $\nabla\cdot(\boldsymbol{a}\times\boldsymbol{h}) = \boldsymbol{h}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{h})$ over the volume V gives the integral identity

$$\int_V \boldsymbol{a}\cdot(\nabla\times\boldsymbol{h})\,\mathrm{d}V = \int_V \boldsymbol{h}\cdot(\nabla\times\boldsymbol{a})\,\mathrm{d}V - \int_V \nabla\cdot(\boldsymbol{a}\times\boldsymbol{h})\,\mathrm{d}V. \tag{2.11}$$

If one identifies $\boldsymbol{a}$ with $\boldsymbol{B}\times\boldsymbol{r}$ and $\boldsymbol{h}$ with the magnetic intensity $\boldsymbol{H}$, the integral in Eq. (2.10) can be written as

$$\int_V \boldsymbol{H}\cdot[\nabla\times(\boldsymbol{B}\times\boldsymbol{r})]\,\mathrm{d}V - \int_S [(\boldsymbol{B}\times\boldsymbol{r})\times\boldsymbol{H}]\cdot\mathrm{d}\boldsymbol{S} , \tag{2.12}$$

where use has been made of the divergence theorem in obtaining the last integral. Using the vector relation $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = (\boldsymbol{a}\cdot\boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c}$, the second integral of Eq. (2.12) can be written as

$$\underset{(1)}{\int_S (\boldsymbol{H}\cdot\boldsymbol{r})\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S}} - \underset{(2)}{\int_S (\boldsymbol{H}\cdot\boldsymbol{B})\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{S}} , \tag{2.13}$$

where the terms to be collected later have been numbered.

The treatment of the first integral in Eq. (2.12) is somewhat more involved. The vector identity

$$\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a})$$

allows this integral to be written as

$$\int_V \boldsymbol{H}\cdot[\nabla\times(\boldsymbol{B}\times\boldsymbol{r})]\,\mathrm{d}V = \underbrace{\int_V \boldsymbol{H}\cdot[(\boldsymbol{r}\cdot\nabla)\boldsymbol{B}]\mathrm{d}V}_{(3)} + 2\int_V \boldsymbol{H}\cdot\boldsymbol{B}\,\mathrm{d}V, \tag{2.14}$$

where $\nabla\cdot\boldsymbol{H} = 0$, $(\boldsymbol{B}\cdot\nabla)\boldsymbol{r} = \boldsymbol{B}$, and $\boldsymbol{B}(\nabla\cdot\boldsymbol{r}) = 3\boldsymbol{B}$ have been used. The first term on the right hand side of Eq. (2.14) can be expanded using the vector identity

$$\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + \boldsymbol{b}\times(\nabla\times\boldsymbol{a}) + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}).$$

This results in

$$\int_V \boldsymbol{H}\cdot[(\boldsymbol{r}\cdot\nabla)\boldsymbol{B}]\mathrm{d}V = \underbrace{\int_V \boldsymbol{H}\cdot[\nabla(\boldsymbol{r}\cdot\boldsymbol{B}) - (\boldsymbol{B}\cdot\nabla)\boldsymbol{r} - \boldsymbol{B}\times(\nabla\times\boldsymbol{r}) - \boldsymbol{r}\times(\nabla\times\boldsymbol{B})]\mathrm{d}V}_{(4)}. \tag{2.15}$$

The second term on the right hand side gives $-\int_V \boldsymbol{H}\cdot\boldsymbol{B}\,\mathrm{d}V$; the third term vanishes; and the last term, using the relation $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c}) = \boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})$ becomes

$$-\int_V \boldsymbol{H}\cdot[\boldsymbol{r}\times(\nabla\times\boldsymbol{B})]\,\mathrm{d}V = \underbrace{-\int_V [(\nabla\times\boldsymbol{B})\times\boldsymbol{H}]\cdot\boldsymbol{r}\mathrm{d}V}_{(5)}. \tag{2.16}$$

Finally, using the relation $\nabla\cdot(\phi\boldsymbol{a}) = \nabla\phi\cdot\boldsymbol{a} + \phi(\nabla\cdot\boldsymbol{a})$, the first term in Eq. (2.15) is

$$\begin{aligned}\int_V \boldsymbol{H}\cdot[\nabla(\boldsymbol{r}\cdot\boldsymbol{B})]\,\mathrm{d}V &= \int_V \left\{\nabla\cdot[(\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{H}] - (\boldsymbol{r}\cdot\boldsymbol{B})\nabla\cdot\boldsymbol{H}\right\}\mathrm{d}V \\ &= \underbrace{\int_S (\boldsymbol{r}\cdot\boldsymbol{B})\boldsymbol{H}\cdot\mathrm{d}\boldsymbol{S}}_{(6)}.\end{aligned} \tag{2.17}$$

Collecting the six numbered terms, the second integral of Eq. (2.9) can be written as

$$\begin{aligned}\int_V [(\nabla\times\boldsymbol{H})\times\boldsymbol{B}]\cdot\boldsymbol{r}\,\mathrm{d}V &= \frac{1}{2}\int_V \boldsymbol{B}\cdot\boldsymbol{H}\,\mathrm{d}V + \int_S (\boldsymbol{r}\cdot\boldsymbol{H})\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \\ &\quad - \frac{1}{2}\int_S (\boldsymbol{H}\cdot\boldsymbol{B})\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{S}.\end{aligned} \tag{2.18}$$

If the field is force-free, the forces of constraint, $\boldsymbol{f}$ in Eq. (2.9), are set equal to zero. This means that the integral on the left hand side of Eq. (2.18) must vanish, which is also identically true since the integrand vanishes in the case of a force-free field. Therefore, the right hand side of Eq. (2.18) requires that

$$\int_V \boldsymbol{B}\cdot\boldsymbol{H}\,\mathrm{d}V = \int_S (\boldsymbol{H}\cdot\boldsymbol{B})\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{S} - 2\int_S (\boldsymbol{r}\cdot\boldsymbol{H})\boldsymbol{B}\cdot\mathrm{d}\boldsymbol{S} \tag{2.19}$$

if the field is to be force-free in V. Thus, since the left hand side of this equation is the total energy, which is positive definite and assumed not to be zero, the field cannot vanish on S: *There must be external forces to balance the outwardly directed pressure due to the total magnetic energy.*

2.1.2. Virial Condition in Terms of the Maxwell Stress Tensor

The Maxwell stress tensor for the magnetostatic field is given by

$$\mu_0 M_{ij} = B_i B_j - \frac{\delta_{ij}}{2} B^2, \tag{2.20}$$

where the Einstein convention of summing over repeated indices is assumed; e.g., $\delta_{ii} = 3$. The total force to be used in Eq. (2.8) is then

$$f_i = f_i + \frac{\partial M_{ij}}{\partial x_j}. \tag{2.21}$$

Substituting Eq. (2.21) into the component form of Eq. (2.8) yields two integrals, the second of which can be written as

$$\int_V \frac{\partial M_{ij}}{\partial x_j} x_i\,\mathrm{d}V = \int_V \left[\frac{\partial}{\partial x_j}(M_{ij}x_i) - M_{ij}\delta_{ij}\right]\mathrm{d}V. \tag{2.22}$$

The first term under the integral on the right hand side is a divergence integrated over the volume, and can be converted by use of Gauss' theorem to a surface integral.

Doing this, and substituting Eq. (2.20) for M_{ij} into the second term, yields for the final form of Eq. (2.8):

$$\frac{1}{2\mu_0}\int_V B^2\,\mathrm{d}V = -\int_V f_i x_i\,\mathrm{d}V - \int_S \mathrm{M}_{ij}x_i\,\mathrm{d}S_j. \tag{2.23}$$

Now if the field is force-free, f_i vanishes, and substituting Eq. (2.20) for the Maxwell stress tensor into the second integral on the right hand side of Eq. (2.23) gives the result

$$\begin{aligned}\int_V \boldsymbol{B}\cdot\boldsymbol{H}\,\mathrm{d}V &= -2\int_S (H_i x_i)B_j\,\mathrm{d}S_j - \frac{2}{\mu_0}\int_S (B_iB_i - \tfrac{\delta_{ii}}{2}B^2)x_j\,\mathrm{d}S_j \\ &= -2\int_S (H_i x_i)B_j\,\mathrm{d}S_j + \int_S (B_iH_i)x_j\,\mathrm{d}S_j.\end{aligned} \tag{2.24}$$

Written in terms of vector inner products, this is the same as Eq. (2.19).

In retrospect, it is clear that using the Maxwell stress tensor to obtain the virial condition is simpler than obtaining it from the field vectors.

2.2 Practical Applications of the Virial Theorem

In terms of practical applications, various coil configurations have been suggested for superconductiong magnetic energy storage,[6] including those that are force-free or, more exactly, force-reduced. In the literature dealing with this subject, it is often stated that the virial theorem imposes the condition

$$m > \frac{\rho}{\sigma}E, \tag{2.25}$$

where m is the structural mass needed to hold the coil together under stress from the magneric forces, ρ is the density of the structural material, σ the average tensile stress, and E the stored energy. Indeed, Eq. (2.25) often appears to be confused with the virial theorem itself. The observation, that the structural mass needed to support a magnetic field producing coil must be proportional to the energy stored, is due to C. Longmire and H. Petschek, and was published by Levy.[7]

Equation (2.25) is somewhat misleading as a general rule, however, since it assumes that separate structural masses are used to maintain the stresses in the principal directions equal to the magnitude of the working stress. This can be seen by observing that since the i-th component of the force on a volume element $\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$ due to magnetic forces and material stresses is

$$\mathrm{d}F_i = \frac{\partial(M_{ij}+S_{ij})}{\partial x_j}\,\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3, \tag{2.26}$$

where S_{ij} is the material stress tensor, the condition for equilibrium for a volume element is

$$\frac{\partial}{\partial x_j}(M_{ij}+S_{ij}) = 0. \tag{2.27}$$

If the condition for equilibrium is multiplied by x_i and integrated over all space, one obtains another form of the virial equation:

$$\int_V \left(\frac{B^2}{2\mu_0} - S_{ii}\right)\mathrm{d}V = -\int_S (M_{ij}+S_{ij})x_i\,\mathrm{d}S_j, \tag{2.28}$$

where the summation convention should be noted, and this result compared to Eq. (2.23). The surface S can be chosen to be at infinity so that the right hand side vanishes.

Since the stress tensor is symmetric, it may be related to a quadric surface known as the stress quadric, which is an ellipsoid, and if the coordinate axes are taken along the principal axes of this ellipsoid, the stress tensor will have a diagonal form. This is really a consequence of a more general theorem that states that any self adjoint matrix (for real inner product spaces this means the matrix is symmetric) can be transformed by a suitable choice of orthonormal basis into a diagonal matrix. The diagonal elements are the principal stresses, and the assumption built into Eq. (2.25) is that the principal stresses are supported by three structural elements, one in each principal direction.

The total structural mass is then determined as follows: In the x_1 direction, for example, the force is $S_{11}\mathrm{d}x_2\mathrm{d}x_3$. Given a maximum tensile stress σ, the cross section is $(S_{11}/\sigma)\mathrm{d}x_2\mathrm{d}x_3$, so that the volume element is $(S_{11}/\sigma)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$. The total mass of structural material is then

$$m = \frac{\rho}{\sigma}\int_V (|S_{11}|+|S_{22}|+|S_{33}|)\,\mathrm{d}V \geq \frac{\rho}{\sigma}\int_V S_{ii}\,\mathrm{d}V. \tag{2.29}$$

Comparing this result to Eq. (2.28) results in Eq. (2.25).

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Chapter 3

SOLUTIONS TO THE FORCE-FREE FIELD EQUATIONS

Methods of obtaining solutions to the force-free field equation $\nabla\times\boldsymbol{B} = \alpha\boldsymbol{B}$ fall into two broad classes: those applicable to the case where α is a constant, and those that yield solutions when α is an arbitrary scalar function of position. In the latter case the equations are nonlinear. In both cases the equations are usually intractable unless a high degree of symmetry is imposed. As mentioned in Chapter 1, in the hydrodynamical literature force-free fields with constant α are known as Trkalian fields, while solutions where α is a function of position are termed Beltrami fields. Analogous to force-free magnetic fields where the current is parallel to the field, in hydrodynamics the vorticity is parallel to the velocity. Although the discussion in this chapter is in the context of magnetostatics, the solutions derived have a direct analogue to incompressible and inviscid fluid flow. The restriction to incompressible flow results from the demand that $\nabla\cdot\boldsymbol{v} = 0$, $\boldsymbol{v}$ being the fluid velocity, which corresponds to $\nabla\cdot\boldsymbol{B} = 0$.

Perhaps the simplest example of a constant α solution to the force-free field equations is given in Cartesian coordinates by $\boldsymbol{B} = B_0[\sin\alpha z, \cos\alpha z, 0]$. The field lines, and hence the current, lie in the x,y-plane with the direction of the lines rotating with increasing z. The handedness of the rotation is determined by the sign of α and the period by the absolute value of α. This type of "twisting" will be seen to be characteristic of solutions to the force-free field equations.

Another simple constant α solution is the axially symmetric Lundquist[1] solution shown in Fig. 3.1. The Lundquist solution is readily derived by restricting consideration to fields having the form $\boldsymbol{B} = [0, B_\phi(r), B_z(r)]$. Assuming this form, the force-free relation in cylindrical coordinates yields the two equations

$$\frac{1}{r}\frac{\mathrm{d}(rB_\phi)}{\mathrm{d}r} = \alpha B_z \tag{3.1}$$

and

$$-\frac{dB_z}{dr} = \alpha B_\phi. \tag{3.2}$$

Substituting B_ϕ from Eq. (3.2) into Eq. (3.1) results in

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_z}{dr}\right) + \alpha^2 B_z = 0. \tag{3.3}$$

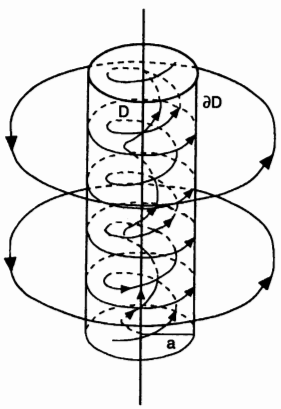

Fig. 3.1. The Lundquist solution. $B_z = J_0(\alpha a) = 0$ on the cylinder $r = a$.

This is Bessel's equation for $J_0(\alpha r)$. Equation (3.2) then determines B_ϕ as $J_1(\alpha r)$. As shown in Fig. 3.1, if one chooses to apply the solution $\boldsymbol{B} = A_0[0, J_1(\alpha r), J_0(\alpha r)]$ in a cylindrical region D bounded by ∂D, such that $J_0(\alpha a) = 0$, the solution matches smoothly to an external field given by $\boldsymbol{B} = [0, (aA_0/r)J_1(\alpha a), 0]$; no surface currents are required.

The constraint $\boldsymbol{B} = [0, B_\phi(r), B_z(r)]$ is quite severe, and one might ask what happens if it is relaxed. When doing so, it rapidly becomes apparent that finding solutions to the constant α field equations becomes much more involved. This can be seen by considering, for example, the case where the field components are allowed to be functions of r and z, and B_r is not required to vanish. Then the requirement that the divergence of $\boldsymbol{B}$ vanish gives

$$\frac{1}{r}\frac{\partial(rB_r)}{\partial r} + \frac{\partial B_z}{\partial z} = 0, \tag{3.4}$$

while the force-free condition results in the three equations:

$$\frac{1}{r}\frac{\partial(rB_\phi)}{\partial r} = \alpha B_z, \tag{3.5}$$

$$-\frac{1}{r}\frac{\partial(rB_\phi)}{\partial z} = \alpha B_r, \tag{3.6}$$

$$\frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} = \alpha B_\phi. \tag{3.7}$$

By differentiating Eq. (3.4) with respect to z and multiplying Eq. (3.7) by r followed by differentiation with respect to r, and subtracting the two resulting equations, one obtains

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_z}{\partial r}\right) + \alpha^2 B_z + \frac{\partial^2 B_z}{\partial z^2} = 0. \tag{3.8}$$

The last term on the left hand side is a consequence of Eq. (3.5). This equation is easily solved for the case where B_z has the very special form $B_z = f(z)g(r)$. The equation can then be separated in terms of a constant of separation λ^2 into two ordinary differential equations

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dg}{dr}\right) + (\alpha^2 - \lambda^2)g = 0 \tag{3.9}$$

and

$$\frac{d^2 f}{dz^2} + \lambda^2 f = 0. \tag{3.10}$$

It will be seen shortly that λ must be pure imaginary. Equation (3.9) is again Bessel's equation and has the solution $g(r) = J_0[(\alpha^2 - \lambda^2)^{1/2} r]$, while Eq. (3.10) has the solution $f(z) = e^{i\lambda z}$. For both solutions there is an arbitrary constant of integration which will be chosen later. Thus, B_z can be written as

$$B_z(r, z) = e^{i\lambda z} J_0[(\alpha^2 - \lambda^2)^{1/2} r]. \tag{3.11}$$

Substitution of Eq. (3.11) into Eq. (3.5) results in

$$\frac{1}{r}\frac{\partial(rB_\phi)}{\partial r} = \alpha e^{i\lambda z} J_0[(\alpha^2 - \lambda^2)^{1/2} r]. \tag{3.12}$$

This equation is readily integrated using the Bessel function relation

$$\int_0^z t^\nu J_{\nu-1}(t)\, dt = z^\nu J_\nu(z), \tag{3.13}$$

and yields

$$B_\phi(r, z) = \frac{\alpha e^{i\lambda z}}{(\alpha^2 - \lambda^2)^{1/2}} J_1[(\alpha^2 - \lambda^2)^{1/2} r]. \tag{3.14}$$

Equation (3.6) then determines B_r to be

$$B_r(r, z) = \frac{-i\lambda e^{i\lambda z}}{(\alpha^2 - \lambda^2)^{1/2}} J_1[(\alpha^2 - \lambda^2)^{1/2} r]. \tag{3.15}$$

It can be verified by direct substitution that Eqs. (3.11), (3.14), and (3.15) satisfy both the divergence and force-free conditions. For the solution to be real, λ must be pure imaginary as mentioned earlier. Let $\lambda \rightarrow i\lambda$, and choose the arbitrary constants of integration to be $(\alpha^2 + \lambda^2)^{1/2}$. The solution then becomes

$$B_z(r, z) = (\alpha^2 + \lambda^2)^{1/2} e^{-\lambda z} J_0[(\alpha^2 + \lambda^2)^{1/2} r], \tag{3.16}$$

$$B_\phi(r, z) = \alpha\, e^{-\lambda z} J_1[(\alpha^2 + \lambda^2)^{1/2} r], \tag{3.17}$$

and

$$B_r(r, z) = \lambda\, e^{-\lambda z} J_1[(\alpha^2 + \lambda^2)^{1/2} r]. \tag{3.18}$$

This rather peculiar looking solution is in fact a combination of the "square toroid" solutions discussed in Section 3.2.3.1. For the solution to be finite, the product λz must always be positive.

The following sections explore various approaches to obtaining solutions to the force-free field equations. Perhaps the most general method, which, however, is only applicable to constant α fields, makes use of solutions to the wave equation. This approach is discussed in Section 3.1. A more direct method of solution, which can be used for non-constant α fields, is to impose various symmetry constraints and attempt to separate the resulting equations in terms of an arbitrary function of either the flux function or the coordinates. This is done in both spherical and cylindrical coordinates in Section 3.2. The axially symmetric Lundquist solution is found to be the simplest member of an hierarchical set of non-constant α solutions.

The case of toroidal coordinates is in a class of its own. No solution is known for non-constant α fields. The equations may be brought into a form reminiscent of the Bernoulli equation, and while this is highly suggestive, the resulting equation appears to be completely intractable. When α is restricted to being a constant, the equation is simplified, but nevertheless does not appear to be directly soluble.

The restriction of α to a constant, however, allows the technique of finding solutions from the wave equation to be used. But now there is the additional problem that the Helmholtz equation is not separable in toroidal coordinates. These coordinates belong to the class of confocal cyclides and the Helmholtz equation only separates in systems of confocal quadrics. [A discussion of the separability of the Helmholtz equation as well as an introduction to the cyclides is given in Appendix 3.] Fortunately, in cylindrical and rotational coordinate systems one variable can be separated out of the Helmholtz equation. In toroidal coordinates, the remaining second order partial differential equation is reducible to a recurrence set of ordinary differential equations which are soluble by known techniques. Solutions in toroidal coordinates are discussed in Section 3.3.

Of course, obtaining a solution to any differential equation is only part of the problem. One must also impose the appropriate boundary conditions. This is far simpler in the case of the scalar fields than it is for vector fields. In Section 3.1.2 three independent solutions to the vector wave equation are introduced. These can be generalized for the purpose of imposing boundary conditions on vector fields. A brief discussion of the issue of boundary conditions as a whole is given in Appendix 3.

3.1 Generation of Solutions to the Force-Free Field Equations from Solutions to the Wave Equation

The force-free field equations with constant α and the wave equation are closely related. In general, all solutions to the constant α force-free field equations are solutions to the wave equation, but the converse is false. The situation is somewhat reminiscent of the relation between Maxwell's equations and the wave equation: while the wave equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ are a necessary consequence of Maxwell's equations, not all solutions to the wave equation satisfy Maxwell's equations.

In the case of force-free fields, taking the curl of the field returns the field multiplied by a constant. While the curl of a solution to the wave equation may not return the same field, adding the solution to its own curl will be seen to guarantee that this will be the case.[2]

For convenience, the discussion will be divided into the "vector" wave equation and the "scalar" wave equation, or Helmholtz equation. While the latter approach to obtaining solutions to the constant α force-free field equations is actually subsumed under the former, it may appear somewhat surprising that the scalar wave equation approach is quite general. The reasons for this, and the relation between the two approaches, are given at the end of Section 3.1.2.

Three related topics are of interest: Ballabh's generalization of the constant α force-free field equations, the generation of electromagnetic fields with $\boldsymbol{E}$ parallel to $\boldsymbol{B}$ from magnetostatic solutions of the force-free field equations, and the relation of the force-free condition to the propagation of circularly polarized waves in optically active media. The first topic is treated in Section 3.1.3 and the second and third in Appendix 1.

3.1.1 The "Vector" Wave Equation

Let $\boldsymbol{H}$ be a solenoidal field ($\nabla \cdot \boldsymbol{H} = 0$) and satisfy as well the force free condition $\nabla \times \boldsymbol{H} = \alpha \boldsymbol{H}$. Consider the field

$$\boldsymbol{H} = \alpha(\nabla \times \boldsymbol{H}') + \nabla \times (\nabla \times \boldsymbol{H}'). \tag{3.19}$$

The requirement the $\boldsymbol{H}$ satisfy the force-free condition will be shown to imply that $\boldsymbol{H}'$ satisfies the wave equation $\nabla^2 \boldsymbol{H}' + \alpha^2 \boldsymbol{H}' = 0$. Assume for the moment that $\boldsymbol{H}'$ does satisfy the wave equation, then Eq. (3.19) can be written as

$$\boldsymbol{H} = \alpha(\nabla \times \boldsymbol{H}') + \alpha^2 \boldsymbol{H}', \tag{3.20}$$

where the vector relation $\nabla^2 \boldsymbol{H}' = \nabla(\nabla \cdot \boldsymbol{H}') - \nabla \times (\nabla \times \boldsymbol{H}')$ has been used. Since $\boldsymbol{H}$ was assumed to satisfy the force-free relation, the form of the right hand side of Eq. (3.20)

shows that adding the curl of a solution to the wave equation to the solution itself generates a solution to the force-free field equations provided the constants are properly arranged.

To see that $\boldsymbol{H}'$ is a solution to the wave equation, take the curl of Eq. (3.19) and set it equal to $\alpha\boldsymbol{H}$ to obtain

$$\alpha\nabla\times(\nabla\times\boldsymbol{H}') + \nabla\times[\nabla\times(\nabla\times\boldsymbol{H}')] = \alpha^2\nabla\times\boldsymbol{H}' + \alpha\nabla\times(\nabla\times\boldsymbol{H}'). \tag{3.21}$$

Simplifying, and expanding the second term on the left hand side, results in

$$\nabla\times[\nabla(\nabla\cdot\boldsymbol{H}') - \nabla^2\boldsymbol{H}'] - \alpha^2\nabla\times\boldsymbol{H}' = 0. \tag{3.22}$$

Since the curl of the first term in the brackets vanishes, Eq. (3.22) becomes

$$\nabla\times[\nabla^2\boldsymbol{H}' + \alpha^2\boldsymbol{H}'] = 0, \tag{3.23}$$

so that

$$\nabla^2\boldsymbol{H}' + \alpha^2\boldsymbol{H}' = \nabla\chi. \tag{3.24}$$

The solution to an equation $\nabla^2\boldsymbol{F} + \alpha^2\boldsymbol{F} = \nabla\beta$ having the form of Eq. (3.24) is $\boldsymbol{F} = \boldsymbol{H}' + \nabla\gamma$, where $\boldsymbol{H}'$ satisfies $\nabla^2\boldsymbol{H}' + \alpha^2\boldsymbol{H}' = 0$ and γ satisfies $\nabla^2\gamma + \alpha^2\gamma = \beta$. Therefore, the arbitrary gradient term in Eq. (3.24) only contributes another gradient to the solution of Eq. (3.24), which makes no contribution because of the curl operations in Eq. (3.19). Thus, if $\boldsymbol{H}$ given by Eq. (3.19) satisfies the constant α force-free field equations, then $\boldsymbol{H}'$ must satisfy the wave equation. Conversely, if $\boldsymbol{H}'$ satisfies the wave equation, then the representation of $\boldsymbol{H}$ given by Eq. (3.19) will satisfy the constant α force-free field equations.

3.1.2 The "Scalar" Wave Equation

If Ψ satisfies the Helmholtz equation

$$\nabla^2\Psi + \alpha^2\Psi = 0, \tag{3.25}$$

then the field given by

$$\boldsymbol{H} = \alpha\nabla\times\Psi\boldsymbol{a} + \nabla\times(\nabla\times\Psi\boldsymbol{a}), \tag{3.26}$$

where $\boldsymbol{a}$ is an arbitrary constant unit vector, will satisfy the constant α force-free field equations.[3]

The proof is straightforward and is similar to that of the last section. Assume the assertion is true. Then for Eq. (3.26) to satisfy the force-free field equations, $\Psi\boldsymbol{a}$ must satisfy

$$\alpha\nabla\times(\nabla\times\Psi\boldsymbol{a}) + \nabla\times[\nabla\times(\nabla\times\Psi\boldsymbol{a})] = \alpha^2\nabla\times\Psi\boldsymbol{a} + \alpha\nabla\times(\nabla\times\Psi\boldsymbol{a}). \tag{3.27}$$

Simplifying results in

$$\nabla\times[\nabla\times(\nabla\times\Psi\boldsymbol{a})] - \alpha^2\nabla\times\Psi\boldsymbol{a} = 0. \tag{3.28}$$

The first term on the right hand side of this equation is

$$\nabla\times[\nabla(\nabla\cdot\Psi\boldsymbol{a}) - \nabla^2\Psi\boldsymbol{a}] = -\nabla\times[\nabla^2(\Psi\boldsymbol{a})]. \tag{3.29}$$

Equation (3.28) can then be written as

$$\nabla\times[\nabla^2(\Psi\boldsymbol{a}) + \alpha^2(\Psi\boldsymbol{a})] = 0. \tag{3.30}$$

This is identically satisfied if Ψ satisfies the Helmholtz equation (since $\boldsymbol{a}$ is a constant vector, if Ψ is a solution to the Helmholtz equation so is $\Psi\boldsymbol{a}$). Conversely, had it been assumed that Eq. (3.26) satisfied the constant α force-free field equations, then Eq. (3.30) would imply that $\nabla^2(\Psi\boldsymbol{a}) + \alpha^2(\Psi\boldsymbol{a}) = \nabla\chi$. For the same reasons as in the previous section, the arbitrary gradient can be set equal zero without loss of generality, thereby implying that Ψ satisfies the Helmholtz equation.

The reasons why Eq. (3.26) has the same generality as Eq. (3.19) can be explained by following the derivation of Eq. (3.26) given by Chandrasekhar and Kendall.[4] If Ψ satisfies the Helmholtz equation $\nabla^2\Psi + \alpha^2\Psi = 0$, then it is easily proved that

$$\boldsymbol{L} = \nabla\Psi, \quad \boldsymbol{T} = \nabla\times(\boldsymbol{a}\Psi), \quad \text{and} \quad \boldsymbol{S} = \frac{1}{\alpha}\nabla\times\boldsymbol{T} \tag{3.31}$$

constitute three independent solutions of the vector wave equation $\nabla^2\boldsymbol{H} + \alpha^2\boldsymbol{H} = 0$. By direct substitution, it is also readily proved that $\nabla\times\boldsymbol{S} = \alpha\boldsymbol{T}$. Therefore,

$$\nabla\times(\boldsymbol{S} + \boldsymbol{T}) = \alpha(\boldsymbol{S} + \boldsymbol{T}). \tag{3.32}$$

Since this is the force-free field equation, the most general solution can be written as

$$\boldsymbol{H} = \boldsymbol{S} + \boldsymbol{T} = \frac{1}{\alpha}\nabla\times[\nabla\times(\boldsymbol{a}\,\Psi)] + \nabla\times(\boldsymbol{a}\,\Psi), \tag{3.33}$$

which is equivalent to Eq. (3.26). Chandrasekhar and Kendall also showed that given any poloidal field $\boldsymbol{S}$, there is a toroidal field, $\boldsymbol{T}$, such that $\boldsymbol{S}+\boldsymbol{T}$ is force-free. The converse is also true.

The three solutions given by Eqs. (3.31) can be generalized and used for imposing boundary conditions on the vector Helmholtz equation. This is discussed in Appendix 3.

3.1.3 Wave Equation Solutions for Ballabh's Generalization of the Force-Free Field Equations

Consider two fields $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ such that

$$\begin{aligned} \nabla\times\boldsymbol{H}_1 &= \alpha_1\boldsymbol{H}_2 \\ \nabla\times\boldsymbol{H}_2 &= \alpha_2\boldsymbol{H}_1, \end{aligned} \tag{3.34}$$

where α_1 and α_2 are constants and $\nabla\cdot\boldsymbol{H}_1 = \nabla\cdot\boldsymbol{H}_2 = 0$. It is clear that this generalization, due to Ballabh,[5] reduces to the usual force-free case when $\alpha_1 = \alpha_2$ and $\boldsymbol{H}_1 = \boldsymbol{H}_2$. Taking the curl of each of Eqs. (3.34) results in

$$\begin{aligned} \nabla\times(\nabla\times\boldsymbol{H}_1) &= \alpha_1\alpha_2\boldsymbol{H}_1 \\ \nabla\times(\nabla\times\boldsymbol{H}_2) &= \alpha_2\alpha_1\boldsymbol{H}_2. \end{aligned} \tag{3.35}$$

Expanding the left hand side of these equations shows that both $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ must satisfy

$$\nabla^2\boldsymbol{H} + \alpha_1\alpha_2\boldsymbol{H} = 0. \tag{3.36}$$

One way to guarantee that the solenoidal condition $\nabla\cdot\boldsymbol{H} = 0$ is met is to set $\boldsymbol{H} = \nabla\times\boldsymbol{F}$. But doing this means that Eq. (3.36) becomes

$$\nabla\times(\nabla^2\boldsymbol{F} + \alpha_1\alpha_2\boldsymbol{F}) = 0, \tag{3.37}$$

or,

$$\nabla^2\boldsymbol{F} + \alpha_1\alpha_2\boldsymbol{F} = \nabla\beta. \tag{3.38}$$

Satisfying the solenoidal condition by setting $\boldsymbol{H} = \nabla\times\boldsymbol{F}$ thus results in the introduction of an arbitrary gradient. It is again easy to show that this gradient will not make a contribution: The solution to Eq (3.38) is

$$\boldsymbol{F} = \boldsymbol{H} + \nabla\gamma, \tag{3.39}$$

where $\boldsymbol{H}$ satisfies

$$\nabla^2\boldsymbol{H} + \alpha_1\alpha_2\boldsymbol{H} = 0, \tag{3.40}$$

and γ satisfies

$$\nabla^2\gamma + \alpha_1\alpha_2\gamma = \beta. \tag{3.41}$$

Since the gradient in Eq. (3.39) drops out in $\boldsymbol{H} = \nabla\times\boldsymbol{F}$, only the condition imposed by Eq. (3.40) is relevant. So if $\boldsymbol{H}$ satisfies Eq. (3.40), and $\boldsymbol{H}_1$ is chosen to be

$$\boldsymbol{H}_1 = \alpha_1\nabla\times\boldsymbol{H}, \tag{3.42}$$

then the first of Eqs. (3.34) determines $\boldsymbol{H}_2$ to be

$$\boldsymbol{H}_2 = \nabla\times(\nabla\times\boldsymbol{H}). \tag{3.43}$$

Equations (3.42) and (3.43) are the general solutions of Eqs. (3.34). That Eqs. (3.42) and (3.43) satisfy Eqs. (3.34) can be seen by direct substitution for the first of Eqs. (3.34), while the second of Eqs. (3.34) is satisfied in light of Eq. (3.40).

3.2 Non-Constant α Solutions to the Force-Free Field Equations in Spherical and Cylindrical Coordinates

When α is not a constant, the technique of finding solutions to the force-free field equation by solving the scalar or vector wave equation is generally not applicable. An alternative approach is to express the magnetic field components in terms of a flux function Ψ (comparable to the stream function in hydrodynamics) and then demand that the field satisfy the force-free condition. The equation that results is a form of the Grad-Shafranov equation[6,7] for constant fluid pressure. The latter is a second order, non-linear potential equation containing a single arbitrary function.

The Grad-Shafranov equation can then separated in terms of a second arbitrary function $g(\Psi)$. This results in a form of Bernoulli's equation for the function α and an equation for the flux function Ψ, which may be solved for the case where $g(\Psi)$ takes a particularly simple form.[8] The solutions to these equations, in both spherical and cylindrical coordinates, will turn out to be generalizations of the solutions given by Chandrasekhar[9], Furth, et al.,[10] and Emets and Zamidra.[11]

It is interesting to note that the introduction of a flux function Ψ is equivalent to writing the field in terms of two functions P and T that respectively generate the poloidal and toroidal components of the field. This is discussed at the end of the next section. It will also be seen in Section 3.2.3.2 that it may sometimes be advantageous to work directly in terms of the field components rather than Ψ, P or T.

3.2.1 Flux Functions, Toroidal and Poloidal Fields

When the magnetic field is symmetric about an axis, one may introduce the analogue of Stokes' stream function for incompressible, axially-symmetric velocity fields. Such a function is known as the flux function,[12] and is introduced as follows: In arbitrary curvilinear coordinates, the divergence of $\boldsymbol{B}$ is given by

$$\nabla\cdot\boldsymbol{B} = \frac{1}{\sqrt{g}}\partial_{x_i}(\sqrt{g}\,B^i), \tag{3.44}$$

where $g = |g_{ij}|$, $i = 1,2,3$. If $\boldsymbol{B}$ is independent of one coordinate (say x_3) the equation

$$\nabla\cdot\boldsymbol{B} = \frac{1}{\sqrt{g}}\partial_{x_1}(\sqrt{g}\,B^1) + \frac{1}{\sqrt{g}}\partial_{x_2}(\sqrt{g}\,B^2) = 0 \tag{3.45}$$

is obviously satisfied if one introduces a function Ψ, such that

$$\sqrt{g}\,B^1 = \partial_{x_2}\Psi, \qquad \sqrt{g}\,B^2 = -\partial_{x_1}\Psi. \tag{3.46}$$

For orthogonal coordinates, $g^{ik} = g_{ik} = 0$ for $i \neq k$; the metric coefficients h_i are defined by $g_{ii} = h_i^2$ and $g^{ii} = 1/h_i^2$; $\sqrt{g} = h_1h_2h_3$. The physical components[13] are then h_iB^i. From Eq. (3.46) these can be written as

$$B_{(1)} = \frac{h_1}{\sqrt{g}}\partial_{x_2}\Psi = \frac{1}{h_2h_3}\partial_{x_2}\Psi, \qquad B_{(2)} = -\frac{h_2}{\sqrt{g}}\partial_{x_1}\Psi = -\frac{1}{h_1h_3}\partial_{x_1}\Psi, \tag{3.47}$$

where the subscript in parentheses designates the physical component (the parentheses will henceforth be dropped).

In spherical coordinates, $(x_1, x_2, x_3) \rightarrow (r, \theta, \phi)$ and $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$. Thus, assuming axial symmetry so that the fields are independent of ϕ,

$$B_r = \frac{1}{r^2\sin\theta}\partial_\theta\Psi, \qquad B_\theta = -\frac{1}{r\sin\theta}\partial_r\Psi. \tag{3.48}$$

In cylindrical coordinates, $(x_1, x_2, x_3) \rightarrow (r, \phi, z)$ and $h_1 = h_3 = 1$, $h_2 = r$. Consequently, for axial symmetry,

$$B_r = -\frac{1}{r}\partial_z\Psi, \qquad B_z = \frac{1}{r}\partial_r\Psi. \tag{3.49}$$

As was mentioned earlier, the introduction of a flux function Ψ is equivalent to writing the field in terms of two functions P and T that respectively generate the poloidal and toroidal components of the field. This can be seen in spherical coordinates, for example, by expanding

$$\boldsymbol{H} = \hat{z}\times\boldsymbol{r}\,T(r,\theta) + \nabla\times\left[\hat{z}\times\boldsymbol{r}\,P(r,\theta)\right] \tag{3.50}$$

to obtain

$$\boldsymbol{H} = \frac{1}{r^2\sin\theta}\partial_\theta[r^2\sin\theta\,P(r,\theta)]\,\hat{r} - \frac{1}{r\sin\theta}\partial_r[r^2\sin\theta\,P(r,\theta)]\,\hat{\theta} + rT(r,\theta)\,\hat{\phi}. \tag{3.51}$$

Defining the flux function as $r^2\sin\theta\,P(r,\theta)$ gives the same result as Eq. (3.48). In this formulation B_ϕ is given by rT.

3.2.2 Spherically Symmetric Solutions

Substituting the magnetic field components, in the form given by Eq. (3.48), into the force-free condition $\nabla\times\boldsymbol{B} = \alpha\boldsymbol{B}$ results in three equations,

$$\frac{\alpha}{r\sin\theta}\partial_\theta\Psi = (\partial_\theta B_\phi + B_\phi\cot\theta), \tag{3.52}$$

$$\frac{\alpha}{r\sin\theta}\partial_r\Psi = (\partial_r B_\phi + \frac{B_\phi}{r}), \tag{3.53}$$

$$\alpha B_\phi = -\frac{1}{r\sin\theta}\Delta^*\Psi, \tag{3.54}$$

where $B_\phi = B_\phi(r, \theta)$ and

$$\Delta^*\Psi = \partial_r^2\Psi + \frac{1}{r^2}\partial_\theta^2\Psi - \frac{1}{r^2}\cot\theta\,\partial_\theta\Psi. \tag{3.55}$$

Eqs. (3.52) and (3.53) together imply that,

$$\partial_r\Psi\,\partial_\theta(r\sin\theta B_\phi) - \partial_\theta\Psi\,\partial_r(r\sin\theta B_\phi) = 0, \tag{3.56}$$

which will be satisfied if $r\sin\theta B_\phi = f(\Psi)$, where $f(\Psi)$ is an arbitrary function of Ψ alone; thus,

$$B_\phi = \frac{1}{r\sin\theta}f(\Psi). \tag{3.57}$$

Equation (3.54) may therefore be written as $\Delta^*\Psi = -\alpha f(\Psi)$. α may now be determined from either Eq. (3.52) or Eq. (3.53) as $\alpha = f'(\Psi)$, where the prime indicates differentiation by Ψ. Defining $\mu = \cos\theta$, Eq. (3.54) may be written in final form as

$$\partial_r^2\Psi + \frac{(1-\mu^2)}{r^2}\partial_\mu^2\Psi = -f(\Psi)f'(\Psi). \tag{3.58}$$

This is a form of the Grad-Shafranov equation. Approximate non-constant α solutions to this equation have been discussed in the astrophysical literature.[14,15] Derivations are given in the following sections of exact non-constant α solutions which are then shown to be generalizations of known solutions.

3.2.2.1 Separation of the Grad-Shafranov Equation in Terms of an Arbitrary Function

Since $f(\Psi)$ is an arbitrary function of Ψ, the right hand side of Eq. (3.58) can be written as $g(\Psi) = -f(\Psi)\alpha$, where $g(\Psi)$ is a second arbitrary function of Ψ. The purpose of this definition is to allow α to be determined in terms of $g(\Psi)$. Differentiating $g(\Psi) = -f(\Psi)\alpha$ with respect to Ψ, one obtains

$$\alpha' - \frac{g'}{g}\alpha - \frac{1}{g}\alpha^3 = 0. \tag{3.59}$$

This is a form of Bernoulli's equation and is completely integrable. Letting $\xi = \alpha^{-2}$, the solution is

$$\xi = A\exp\left(-2\int^{\Psi}\frac{g'}{g}\,d\Psi\right) - 2\exp\left(-2\int^{\Psi}\frac{g'}{g}\,d\Psi\right)\int^{\Psi}\frac{1}{g}\exp\left(2\int^{\Psi}\frac{g'}{g}\,d\Psi\right)d\Psi. \tag{3.60}$$

where A is a constant of integration.

While $g(\Psi)$ is still arbitrary in Eq. (3.60), the remaining equation, $\Delta^*\Psi = g(\Psi)$, will apparently only separate if $g(\Psi) = -\kappa^2\Psi$, where κ is a constant. If $g(\Psi)$ is so restricted and Ψ is assumed to be of the form $\Psi = \Phi(r)\Gamma(\mu)$, the equation separates, in terms of a separation constant λ, into

$$\Phi''(r) + \left(\kappa^2 - \frac{\lambda}{r^2}\right)\Phi(r) = 0 \tag{3.61}$$

and

$$\Gamma''(\mu) + \frac{\lambda}{(1-\mu^2)}\Gamma(\mu) = 0. \tag{3.62}$$

For $g(\Psi) = -\kappa^2\Psi$, Eq. (3.60) gives $\xi = A\,\Psi^{-2} + 1/\kappa^2$ so that α is

$$\alpha = \frac{\kappa\Psi}{(B^2+\Psi^2)^{1/2}}, \qquad B^2 = A\kappa^2. \tag{3.63}$$

Note that if the constant $B = 0$, $\alpha = \kappa$; and since $\alpha = f'(\Psi)$, apart from a constant of integration, $f(\Psi)$ is determined by Eq. (3.63) to be

$$f(\Psi) = \int \alpha\,d\Psi = \kappa(B^2+\Psi^2)^{1/2}. \tag{3.64}$$

Now if $f(\Psi)$ does not have a zero on the polar axis, Eq. (3.57) implies that B_ϕ is singular. Therefore, if the constant B is not required to vanish, the axis must be excluded from the domain.[16] This is a physically reasonable requirement if B is interpreted as being proportional to an axial line current. Because of the symmetry of the problem, such an axial current will only generate a toroidal field, which can flow without altering the force free character of the configuration. While such a line current is mathematically acceptable, real world applications would require the imposition of boundary conditions corresponding to a finite current density.

3.2.2.2 Solution of the Separated Grad-Shafranov Equation

Consider Eq. (3.61). If λ is restricted to the values $\lambda = n(n+1)$, $n = 0, \pm1, \pm2, \ldots$, the solutions to this equation are the Riccati-Bessel functions with argument κr; i.e., $(\kappa r)\,c_n(\kappa r)$ where $c_n(\kappa r)$ is one of $\{j_n(\kappa r), y_n(\kappa r), h_n^{(1)}(\kappa r), h_n^{(2)}(\kappa r)\}$. The properties of these functions follow from those of the spherical Bessel functions.

The solution to Eq. (3.62) can be given in terms of the generalized Jacobi polynomials[17] as

$$(1-\mu)^{(\alpha+1)/2}(1-\mu)^{(\beta+1)/2}P_m^{(\alpha,\beta)}(\mu), \quad \alpha,\beta = \pm1, \quad m = 0, 1, 2, \ldots, \tag{3.65}$$

provided Eq. (3.62) is rewritten as

$$\Gamma''(\mu) + G^{(\alpha,\beta)}(m,\mu)\,\Gamma(\mu) = 0, \tag{3.66}$$

and $G^{(\alpha,\beta)}(m,\mu)$ is restricted to the form:

$\beta \backslash \alpha$	$+1$	-1
$+1$	$\dfrac{m(m+3)+2}{(1-\mu^2)}$	$\dfrac{m(m+1)}{(1-\mu^2)}$
-1	$\dfrac{m(m+1)}{(1-\mu^2)}$	$\dfrac{m(m-1)}{(1-\mu^2)}$

$G^{(1,-1)}$ and $G^{(-1,1)}$ will be compatible with the requirement from Eq. (3.61) that $\lambda = n(n+1)$ if $m = n$. The same is true for $G^{(1,1)}$ if $m = n-1$, and $G^{(-1,-1)}$ if $m = n+1$. (While other possibilities exist for negative n, they add nothing substantive to the discussion.)

Given these constraints, there are apparently four possible solutions given by Eq. (3.65):

$$\Gamma(\mu) = \left\{ \begin{array}{c} (1-\mu)\, P_n^{(1,-1)}(\mu) \\ (1+\mu)\, P_n^{(-1,1)}(\mu) \\ (1-\mu)(1+\mu)\, P_{n-1}^{(1,1)}(\mu) \\ P_{n+1}^{(-1,-1)}(\mu) \end{array} \right\}, \quad n \geq 1 \tag{3.67}$$

The restriction to $n \geq 1$ is imposed to exclude the $n = 0$ case for the first two solutions because these can be ruled out on physical grounds (they require sources and sinks on the negative z-axis); and for $n = 0$, the third and fourth sets of solutions are zero. In the case of the fourth solution, $P_1^{(-1,-1)}(\mu) = 0$ since this polynomial satisfies the condition $n+\alpha+\beta+k = 0$, $\alpha = -j$, $1 \leq k \leq j \leq n$ [See Ref. 17]. Note also that all four possible solutions have zeros at $\mu = \pm 1$, corresponding to the positive and negative z-axis respectively. Each of the solutions of Eq. (3.67) can be combined with a solution of Eq. (3.61) of order n to form the flux function.

The four solutions given by Eqs. (3.67) cannot, however, be linearly independent since Eq. (3.66) is only of second order. It is shown in Appendix 2 that the following relations hold for $n \geq 1$:

$$(1-\mu)P_n^{(1,-1)}(\mu) = -(1+\mu)\, P_n^{(-1,1)}(\mu), \tag{3.68}$$

$$P_{n+1}^{(-1,-1)}(\mu) = \frac{1}{4}(\mu+1)(\mu-1)\, P_{n-1}^{(1,1)}(\mu), \tag{3.69}$$

$$P_{n+1}^{(-1,-1)}(\mu) = \frac{n}{2(n+1)}(\mu+1)\, P_n^{(-1,1)}(\mu), \tag{3.70}$$

$$(1+\mu)P_n^{(-1,1)}(\mu) = -\frac{(n+1)}{2n}(1-\mu)(1+\mu)\, P_{n-1}^{(1,1)}(\mu), \tag{3.71}$$

$$(1-\mu)P_n^{(1,-1)}(\mu) = -\frac{2(n+1)}{n}\, P_{n+1}^{(-1,-1)}(\mu), \tag{3.72}$$

$$(1-\mu)P_n^{(1,-1)}(\mu) = \frac{(n+1)}{2n}\,(1-\mu)(1+\mu)P_{n-1}^{(1,1)}(\mu). \tag{3.73}$$

This set of relations allow any of the four solutions to be written in terms of any other, and the third is of particular interest. The reason is that the latter solution can in turn be easily related to those given by Chandrasekhar [Ref. 9]: In Eq. (3.63), let the constant $B = 0$ so that $\alpha = \kappa$. Remembering that $G^{(1,1)}(m,\mu)$ will be compatible with $c_n(\kappa r)$ if $m = n-1$, the flux function associated with $G^{(1,1)}(m,\mu)$ is given by

$$\Psi = (\alpha r)\, c_m(\alpha r)\, (1+\mu)(1-\mu)P_m^{(1,1)}(\mu). \tag{3.74}$$

The discussion at the end of Section 3.2.1 and the definition of P following Eq. (3.51), combined with the relation

$$C_n^{\beta}(\mu) = \frac{\Gamma(\beta+1/2)\,\Gamma(2\beta+n)}{\Gamma(2\beta)\,\Gamma(\beta+n+1/2)}\, P_n^{(\beta-1/2\,,\,\beta-1/2)}(\mu), \quad \beta \neq 0 \tag{3.75}$$

allow this set of solutions to be written in terms of

$$P^m = \frac{D_{m+3/2}(\alpha r)}{r^{3/2}}\, C_m^{3/2}(\mu). \tag{3.76}$$

Here, $D_{m+3/2}(\alpha r)$ is a general cylinder function of order $m+3/2$ and the $C_m^{3/2}(\mu)$ are the Gegenbauer or ultraspherical polynomials. The function P that generates the poloidal field is given by $P = \sin\theta\, P^m$ When the constant $B \neq 0$, the four sets of solutions given in Eq. (3.67) are therefore seen to be a simple generalization of Chandrasekhar's results.

Other solutions appearing in the literature can also be related to those given here. For example, consider the case where $\alpha = +1$, $\beta = -1$. (As has been shown, the particular case chosen is irrelevant.) Then the flux function, apart from an arbitrary multiplicative constant, is

$$\Psi = (\kappa r)\, c_n(\kappa r)\, (1-\mu)P_n^{(1,-1)}(\mu). \tag{3.77}$$

The magnetic field components are then given by Eqs. (3.48), (3.57) and (3.64) as

$$B_r = \frac{\kappa}{r}\, c_n(\kappa r)\left\{\frac{(n+1)}{n(1+\mu)}\left[\mu n P_n^{(1,-1)}(\mu) - (n-1)P_{n-1}^{(1,-1)}(\mu)\right]\right\}, \tag{3.78}$$

$$B_\theta = \frac{\kappa(1-\mu)}{r\sin\theta} P_n^{(1,-1)}(\mu)\left\{nc_n(\kappa r) - (\kappa r)c_{n-1}(\kappa r)\right\}, \tag{3.79}$$

$$B_\phi = \frac{\kappa}{r\sin\theta}\left\{B^2 + (\kappa r)^2[c_n(\kappa r)]^2(1-\mu)^2[P_n^{(1,-1)}(\mu)]^2\right\}^{1/2}. \tag{3.80}$$

For $c_n(\kappa r) = j_n(\kappa r)$, and $n = 1$, the expression for Ψ becomes,

$$\Psi = \left(\frac{\sin\kappa r}{\kappa r} - \cos\kappa r\right)\sin^2\theta. \tag{3.81}$$

This is the form of the flux function given by Morikawa[18] and also, without the imposition of the boundary condition of a uniform external field, that given by Emets and Zamidra [Ref. 11]. Both Morikawa and Emets and Zamidra only give the lowest order ($n = 1$) solution. Morikawa imposes the boundary condition of a perfectly conducting spherical cavity and assumes that α is constant, while Emets and Zamidra show that the constant B in Eq. (3.63) can be interpreted to be proportional to an axial current. The solutions of Eqs. (3.78), (3.79) and (3.80) are then a generalization of the particular solutions given by these authors.

The character of these solutions is illustrated in Fig. 3.2 for $n = 1,3$. There, the surfaces of constant Ψ are plotted as a function of r and θ (ϕ being assumed constant). Irregularities in the plot should be ignored since they merely reflect limitations in the way the function was sampled for plotting.

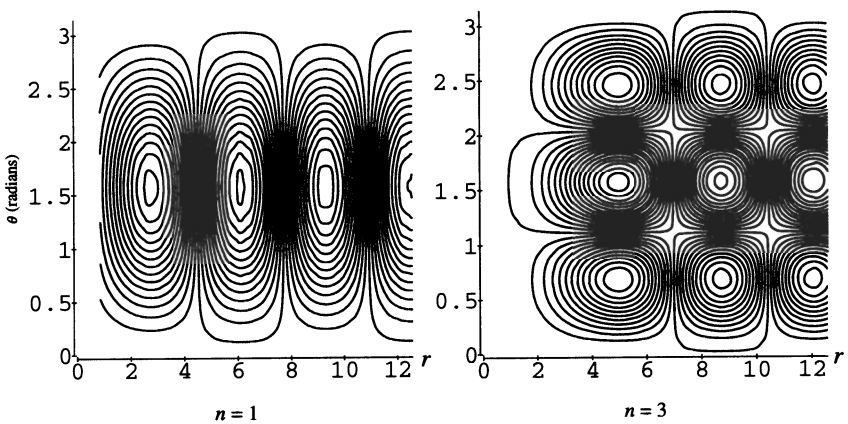

Fig. 3.2 Contour Plot of $\Psi = r\sqrt{\pi/2r}\, J_{n+1/2}(r)(1-\mu)P_n^{(1,-1)}(\mu)$ for $n = 1$ and $n = 3$.

Fig. 3.3 shows the $n = 1$ case for particular values of Ψ as a function of $z = r\cos\theta$ and r. The magnetic surfaces, outside the familiar central one, are toroids with a bananalike cross section.

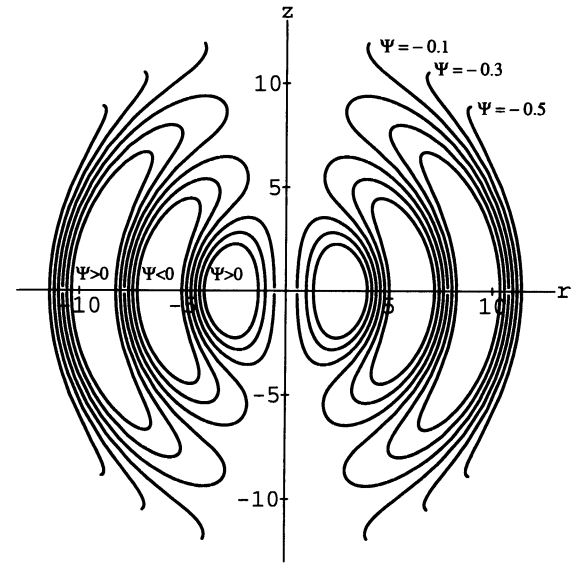

Fig. 3.3. Parametric plot of $\Psi = r\sqrt{\pi/2r}\, J_{n+1/2}(r)(1-\mu)P_n^{(1,-1)}(\mu)$ for $n = 1$ as a function of r and $z = r\cos\theta$. In the figure the contours are for $\Psi = \pm 0.1, \pm 0.3, \pm 0.5$. Note that $\Psi = 0$ on the z-axis.

3.2.3 Axially Symmetric Solutions

For axially symmetric fields such that $\boldsymbol{B} = [0, B_\phi(r), B_z(r)]$, the force-free condition, $\nabla\times\boldsymbol{B} = \alpha\boldsymbol{B}$, can be put in the form

$$\frac{d}{dr}(B_\phi^2 + B_z^2) + 2\frac{B_\phi^2}{r} = 0, \tag{3.82}$$

with α being given by,

$$\alpha(r) = \frac{1}{B_z}\frac{1}{r}\frac{d(rB_\phi)}{dr}. \tag{3.83}$$

Any function $g(r) = B_\phi^2 + B_z^2$ such that $g(r) \geq 0$, $g'(r) \leq 0$, and $\frac{d[r^2 g(r)]}{dr} \geq 0$ will give a non-singular solution of Eq. (3.82). $g(r)$ can be thought of as a generating function.[19] The

condition $g(r) \geq 0$ guarantees that the energy density is positive; $g'(r) \leq 0$ means that the energy density declines monotonically with increasing r; and the last condition implies that $g(r)$ does not fall off faster than $1/r^2$.

3.2.3.1 The Grad-Shafranov Equation for Axial Symmetry

Using the definitions of the field components in Eq. (3.49), and following the same procedures as in the case of spherical coordinates, results in the three equations

$$\partial_r^2\Psi - \frac{1}{r}\partial_r\Psi + \partial_z^2\Psi = -f(\Psi)f'(\Psi), \tag{3.84}$$

$$B_\phi = \frac{1}{r}f(\Psi), \tag{3.85}$$

$$\alpha = f'(\Psi). \tag{3.86}$$

If $g(\Psi)$ is defined as $g(\Psi) = -f(\Psi)f'(\Psi)$, α will again satisfy Bernoulli's equation with the solution given by Eq. (3.60).

As before, Eq. (3.84) will apparently only separate if $g(\Psi) = -\kappa^2\Psi$, yielding

$$\Phi''(r) - \frac{1}{r}\Phi'(r) + \left(\kappa^2 - \lambda^2\right)\Phi(r) = 0, \tag{3.87}$$

$$\Gamma''(z) + \lambda^2\,\Gamma(z) = 0, \tag{3.88}$$

where here $\Psi = \Phi(r)\Gamma(z)$ and λ^2 is again a separation constant. Both equations are immediately solvable, and apart from an arbitrary constant of integration, have the solutions

$$\Phi(r) = r\,D_1\left(\left[\kappa^2-\lambda^2\right]^{1/2}r\right) \tag{3.89}$$

and

$$\Gamma(z) = \left\{\begin{array}{c}\sin\lambda z\\ \cos\lambda z\end{array}\right\}, \tag{3.90}$$

where D_1 is a cylinder function of first order. Here, while the constants of integration and the particular cylinder function may be chosen to fit the boundary conditions of the problem at hand, the order of the cylinder function is fixed.

Defining $\gamma = \left[\kappa^2-\lambda^2\right]^{1/2}$, Eqs. (3.49) and (3.85) give the field components as

$$B_r = \lambda D_1(\gamma r)\left\{\begin{array}{c}-\cos\lambda z\\ \sin\lambda z\end{array}\right\}, \tag{3.91}$$

$$B_z = \frac{1}{r}\left[D_1(\gamma r) + (r\gamma)\partial_{(\gamma r)}D_1(\gamma r)\right]\left\{\begin{array}{c}\sin\lambda z\\ \cos\lambda z\end{array}\right\}, \tag{3.92}$$

$$B_\phi = \frac{\kappa}{r}\left(B^2 + r^2\left[D_1(\gamma r)\right]^2\left\{\begin{array}{c}\sin\lambda z\\ \cos\lambda z\end{array}\right\}^2\right)^{1/2}. \tag{3.93}$$

For the particular case $D_1(\gamma r) = J_1(\gamma r)$, $B = 0$ and the choice of $\cos\lambda z$ in the expression for Ψ, this is the "square toroid" solution given by Furth, et al. [Ref. 10]. The solution given by Eqs. (3.16), (3.17), and (3.18) can be obtained by also choosing $D_1(\gamma r) = J_1(\gamma r)$, $B = 0$, and setting $\gamma = \left(\alpha^2 - \lambda^2\right)^{1/2}$, with $\lambda \to i\lambda$. Letting λ be pure imaginary transforms the trigonometric functions of Eqs. (3.91), (3.92), and (3.93) into hyperbolic functions and an appropriate combination of the resulting solutions yields Eqs. (3.16), (3.17), and (3.18).

3.2.3.2 A Hierarchical Set of Solutions for a special Class of Fields

The approach to axially symmetric fields given in this section is based on that introduced by Lüst and Schlüter [Ref. 19] and used by Chandrasekhar [Ref. 9]. The condition that none of the components of a magnetic field have an azimuthal dependence means that $\boldsymbol{H}$ can be written as a sum of a toroidal and a poloidal field as in Eq. (3.50):

$$\boldsymbol{H} = \hat{z}\times\boldsymbol{r}\,T(r,z) + \nabla\times\left[\hat{z}\times\boldsymbol{r}\,P(r,z)\right]. \tag{3.94}$$

Here the axis of symmetry is along $\hat{z}$ and the functions T and P respectively generate the toroidal and poloidal fields. Expressing the first term on the right hand side of this equation in terms of the azimuthal coordinate $\hat{\phi}$, and expanding the triple vector product in the second term allows the general form for $\boldsymbol{H}$ to be written as

$$\boldsymbol{H} = -r\frac{\partial P(r,z)}{\partial z}\hat{r} + rT(r,z)\,\hat{\phi} + \frac{1}{r}\frac{\partial r^2P(r,z)}{\partial r}\hat{z}. \tag{3.95}$$

Computing $\nabla\times\boldsymbol{H}$ and using $\nabla\times\boldsymbol{H} = \alpha\boldsymbol{H}$, the expressions for the $\hat{r}$ and $\hat{z}$ components combine to give

$$\alpha\nabla(r^2P) = \nabla(r^2T), \tag{3.96}$$

while the $\hat{\phi}$ component yields

$$\frac{\partial^2P}{\partial r^2} + \frac{3}{r}\frac{\partial P}{\partial r} + \frac{\partial^2P}{\partial z^2} = -\alpha T. \tag{3.97}$$

Now one can either use the component form of Eq. (3.96) or $\nabla\alpha\cdot\boldsymbol{H} = 0$ with the general form of $\boldsymbol{H}$ given by Eq. (3.95) to obtain

$$\frac{\partial(r^2P)}{\partial r}\frac{\partial\alpha}{\partial z}-\frac{\partial\alpha}{\partial r}\frac{\partial(r^2P)}{\partial z} = 0. \tag{3.98}$$

This is the same as

$$\nabla(r^2P)\times\nabla\alpha = 0. \tag{3.99}$$

Thus, $\nabla\alpha$ and $\nabla(r^2P)$ are in the same direction, and $\alpha = \alpha(r^2P)$ is only a function of r^2P. By again considering the component form of Eq. (3.96), the relationship between α and r^2T can be written

$$r^2T = \int \alpha(r^2P)\mathrm{d}(r^2P). \tag{3.100}$$

Thus far, the only constraint imposed on the force-free field equations has been that none of the components of the magnetic field have an azimuthal dependence. Consider now the case where $\boldsymbol{H} = [0, H_\phi(r), H_z(r)]$. For this form of $\boldsymbol{H}$, the equations are completely solvable if α is a constant, and solvable in terms of an arbitrary function of r if α is not a constant. In addition, it can be seen from Eq. (3.95) that for this case P and T are only functions of r. The basic equations, Eqs. (3.96), (3.97), and (3.98), then become

$$\alpha\frac{\mathrm{d}(r^2P)}{\mathrm{d}r} = \frac{\mathrm{d}(r^2T)}{\mathrm{d}r}, \tag{3.101}$$

$$\frac{\mathrm{d}^2P}{\mathrm{d}r^2}+\frac{3}{r}\frac{\mathrm{d}P}{\mathrm{d}r} = -\alpha T, \tag{3.102}$$

$$\frac{\mathrm{d}(r^2P)}{\mathrm{d}r}\frac{\mathrm{d}\alpha}{\mathrm{d}z} = 0. \tag{3.103}$$

In the last equation, $\frac{\mathrm{d}\alpha}{\mathrm{d}z} = 0$ since α is a function of r^2P and P is only a function of r.

3.2.3.2.1 Solution for the Case Where $\boldsymbol{H} = [0, H_\phi(r), H_z(r)]$

Consider first the case where α is equal to a constant. Equation (3.100) is then immediately integrable to $rT = \alpha rP$. Substituting for rT in Eq. (3.102) yields

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}(rP)}{\mathrm{d}r}\right)+\left(\alpha^2-\frac{1}{r^2}\right)rP = 0, \tag{3.104}$$

which is Bessel's equation for rP. Apart from a constant, the solution for P and T is $P = J_1(\alpha r)/r$ and $T = \alpha P = \alpha J_1(\alpha r)/r$ respectively. Using this result in the general expression for the magnetic field, Eq. (3.95), gives

$$\boldsymbol{H} = AJ_1(\alpha r)\hat{\phi}+A\,J_0(\alpha r)\hat{z}, \tag{3.105}$$

where A is an arbitrary constant. This is the well known result obtained by Lundquist [Ref. 1].

When α is not a constant, the expression for α given by Eq.(3.101), and the fact that

$$H_\phi = rT \quad \text{and} \quad H_z = \frac{1}{r}\frac{\mathrm{d}(r^2P)}{\mathrm{d}r} \tag{3.106}$$

can be used to rewrite Eq. (3.102) as

$$\frac{1}{H_\phi}\frac{\mathrm{d}H_z}{\mathrm{d}r}+\frac{1}{H_z}\frac{1}{r}\frac{\mathrm{d}(rH_\phi)}{\mathrm{d}r} = 0. \tag{3.107}$$

Building on the work of Alfvén,[20] who introduced a new variable $k = H_\phi/H_z$, Murty[21] obtained the differential equation

$$\frac{1}{H_z}\frac{\mathrm{d}H_z}{\mathrm{d}r}+\frac{1}{(1+k^2)}\frac{k}{r}\frac{\mathrm{d}(kr)}{\mathrm{d}r} = 0. \tag{3.108}$$

Because k is the tangent of the angle between $\boldsymbol{H}$ and $\hat{z}$, finding k and H_z solves the problem. By separating this equation in terms of an arbitrary function of r, Murty found that the solution could be written as

$$H_z = H_0\exp\left(-\int_0^r \Phi(r)\mathrm{d}r\right) \tag{3.109}$$

and

$$k^2 = 2r^{-2}\exp\left(2\int^r \Phi(r)\mathrm{d}r\right)\int_0^r \Phi(r)r^2\exp\left(-2\int^r \Phi(r)\mathrm{d}r\right)\mathrm{d}r \tag{3.110}$$

A somewhat different approach was used by Marsh.[22] Instead of introducing the new variable k, Eq. (3.107) was first multiplied by $H_\phi H_z$ and then separated in terms of an arbitrary function $\varphi(r)$ to obtain

$$H_z\frac{\mathrm{d}H_z}{\mathrm{d}r} = -\varphi(r) \tag{3.111}$$

and

$$H_\phi\frac{1}{r}\frac{\mathrm{d}(rH_\phi)}{\mathrm{d}r} = \varphi(r). \tag{3.112}$$

The first equation can be immediately integrated,

$$H_z^2 = -2\int \varphi(r)\mathrm{d}r + C, \tag{3.113}$$

where C is an arbitrary constant of integration.

By defining a new variable, u, such that $H_\phi = u^{-1}$, the second equation becomes

$$\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{u}{r} + u^3\varphi(r) = 0. \tag{3.114}$$

This is Bernoulli's equation. [The equation that results from separating Eq. (3.108) in terms of an arbitrary function $\Phi(r)$ also results in a form of Bernoulli's equation.] It is reducible to the standard form of a linear equation of the first order, and has the solution

$$H_\phi^2 = Ar^{-2} + 2r^{-2}\int \varphi(r)r^2\mathrm{d}r, \tag{3.115}$$

where A is an arbitrary constant of integration which is henceforth set equal to zero to guarantee that H_ϕ is finite on the z-axis (for the same reason, the constant associated with the integration on the right hand side of the equation is also set equal to zero). The solution given here, and that of Murty, are clearly related by a simple transformation.

The problem has thus been reduced to finding those functions $\varphi(r)$ that correspond to physically interesting results.

3.2.3.2.2 An Hierarchical Set of Solutions

If Eqs. (3.113) and (3.115) are differentiated, and the components of $\boldsymbol{H}$ set equal to the constant α values in Eq. (3.105), it is immediately apparent that $\varphi(r) = J_0(r)J_1(r)$. This suggests that functions of the form

$$\varphi(r) = \sum_{n,m=0}^{\infty} a_{nm}\, C_n(r)\, D_m(r), \tag{3.116}$$

where C_n and D_m are cylinder functions, and the a_{nm} are appropriately chosen constants, may be of interest. Such functions lead to integrals of the form

$$\int C_n(r)\, D_m(r)\, \mathrm{d}r \quad \text{and} \quad \int r^2\, C_n(r)\, D_m(r)\, \mathrm{d}r$$

when determining H_z and H_ϕ. While such integrals are in general quite difficult to evaluate, the following two expressions given by Luke[23] allow the integrals to be evaluated for special cases:

$$\int^z J_\mu(t)\, J_{\mu+1}(t)\, \mathrm{d}t = \sum_{k=0}^{\infty} J_{\mu+k+1}^2(z), \tag{3.117}$$

$$\begin{aligned}(\rho+\mu+\nu)\int^z t^{\rho-1}\, C_\mu(t)\, D_\nu(t)\, \mathrm{d}t + (\rho-\mu-\nu-2)\int^z t^{\rho-1}\, C_{\mu+1}(t)\, D_{\nu+1}(t)\, \mathrm{d}t \\ = z^\rho\left(C_\mu(z)\, D_\nu(z) + C_{\mu+1}(z)\, D_{\nu+1}(z)\right).\end{aligned} \tag{3.118}$$

Setting $\rho = 3$ in the second equation gives

$$\begin{aligned}(3+\mu+\nu)\int^z t^2\, C_\mu(t)\, D_\nu(t)\, \mathrm{d}t + (1-\mu-\nu)\int^z t^2\, C_{\mu+1}(t)\, D_{\nu+1}(t)\, \mathrm{d}t \\ = z^3\left(C_\mu(z)\, D_\nu(z) + C_{\mu+1}(z)\, D_{\nu+1}(z)\right).\end{aligned} \tag{3.119}$$

Restricting consideration henceforth to Bessel functions of the first kind, and of integer order (relabel μ by m and ν by n) with real arguments, there are two cases that *prima facie* appear to be promising: $n = m+1$ and $n = -m+1$.

Case 1: $n = m+1$

Substituting $n = m+1$ into Eq. (3.119), it can be seen that an appropriate choice for the arbitrary separation function in Eqs. (3.111) and (3.112) is

$$\varphi(r) = \frac{1}{2}\left[(m+2)J_m(r)J_{m+1}(r) - mJ_{m+1}(r)J_{m+2}(r)\right]. \tag{3.120}$$

The integral of Eq. (3.115) is straightforward and gives

$$H_\phi^2 = (m+1)J_{m+1}^2(r). \tag{3.121}$$

The right hand side is positive definite for all values of r, so that H_ϕ is well defined. Using Eq. (3.117), the integral of Eq. (3.113) is

$$H_z^2 = -(m+2)\sum_{k=0}^{\infty} J_{m+k+1}^2(r) + m\sum_{k=0}^{\infty} J_{m+k+2}^2(r) + C. \tag{3.122}$$

Using the relation

$$J_0^2(r) + 2\sum_{k=1}^{\infty} J_k^2(r) = 1, \tag{3.123}$$

H_z^2 can be written as

$$H_z^2 = -1 + J_0^2(r) + 2\sum_{k=1}^{m} J_k^2(r) - mJ_{m+1}^2(r) + C. \tag{3.124}$$

If $m = 0$, this solution will match the constant α solution of Lundquist if $C = 1$. However, setting $C = 1$ will allow H_z^2 to be negative for some values of r, thus making H_z pure imaginary for these values. Thus, while the solutions corresponding to $n = m+1$ may be of some intrinsic interest, they cannot be used to form a hierarchical set that reduces to the Lundquist solution for the lowest value of m.

Case 2: $n = -m+1$

Proceeding in the same manner, substitution of $n = -m+1$ into Eq. (3.119) suggests that $\varphi(r)$ be chosen as

$$\varphi(r) = J_m(r)J_{m-1}(r). \tag{3.125}$$

The constant α case then corresponds to $m = 1$. Substitution of Eq. (3.125) into Eqs. (3.113) and (3.115) yields, setting the constant of integration again equal to unity,

$$\begin{aligned} H_z^2 &= J_0^2(r), & m &= 1, \\ H_z^2 &= J_0^2(r) + 2\sum_{k=1}^{m-1} J_k^2(r) & m &> 1, \end{aligned} \tag{3.126}$$

and

$$H_\phi^2 = \frac{r}{2}\left[J_m(r)J_{m-1}(r) - J_{m+1}(r)J_{m-2}(r)\right]. \tag{3.127}$$

Both Eqs. (3.126) and (3.127) are positive over the whole range of r, and therefore constitute a well defined set of solutions to the force-free field equations that reduce to the Lundquist solution for $m = 1$.

An expression for the function $\alpha(r)$ can readily be obtained from Eqs. (3.101) and (3.106),

$$\alpha(r) = \frac{1}{H_z}\frac{1}{r}\frac{d(rH_\phi)}{dr}. \tag{3.128}$$

Or, using Eqs. (3.113) and (3.115),

$$\alpha(r) = \frac{\varphi(r)}{H_z H_\phi}. \tag{3.129}$$

Substituting from Eqs. (3.125), (3.126) and (3.127),

$$\alpha(r) = \frac{J_m(r)J_{m-1}(r)}{\left[\frac{r}{2}\left(J_m(r)J_{m-1}(r) - J_{m+1}(r)J_{m-2}(r)\right)\right]^{1/2}\left[J_0^2(r) + 2\sum_{k=1}^{m-1} J_k^2(r)\right]^{1/2}}, \quad m > 1. \tag{3.130}$$

For $m = 1$, Eq. (3.129) gives a constant value, $\alpha(r) = 1$, as expected.

The general behavior of $H_z(r)$, $H_\phi(r)$ and $\alpha(r)$ is shown in Fig. 3.4. The angle between $\boldsymbol{H}$ and the z-axis is given by $\tan^{-1}(H_\phi/H_z)$ and is shown in Fig. 3.5. Note that the behavior shown in Fig. 3.5 is significantly different than that of the Lundquist solution given by $m = 1$.

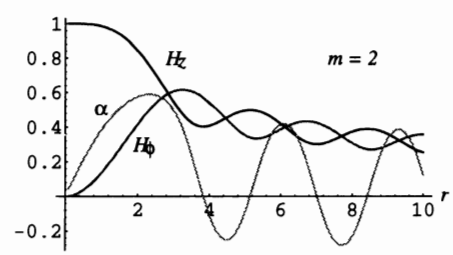

Fig. 3.4. Magnetic field components $H_z(r)$, $H_\phi(r)$, and the function $\alpha(r)$ for $m = 2$.

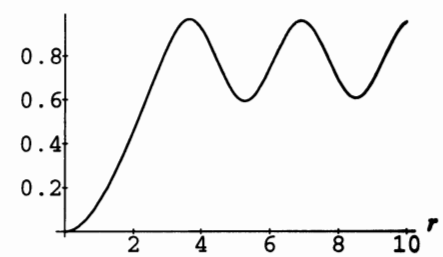

Fig. 3.5. The angle $\tan^{-1}[H_\phi(r)/H_z(r)]$ between $\boldsymbol{H}$ and the z-axis for m=2.

3.3 Solutions in Toroidal Coordinates

Solutions to the force-free field equations in toroidal coordinates, while they may be of great interest from an applied perspective, are also the most difficult to obtain. An investigation into these solutions, for the purpose of producing large magnitude magnetic fields exterior to a current carrying toroid, was initiated by Buck.[24] He solved the boundary value problem given in toroidal coordinates by $B_\mu(\mu_0) = 0$ (i.e., that the normal component of the field vanish at the surface of the torus). His approach was to use Stokes'

current function to find an exact solution for the region exterior to the toroid, and Galerkin's method to find an approximate solution for the region interior to the toroid. For the exterior region, the problem is analogous to that of a solid torus moving through a liquid. An approximation method was used for the interior region because of the difficulty in satisfying the boundary conditions.

Tsuji[25] found a set of analytic solutions for the interior region of the toroid that are derived from the Helmholtz equation. In terms of a fundamental set of solutions S_l, he found that the force-free solution for the toroidal component of the field interior to the torus is given by

$$h_\phi B_\phi = \sum_{l=0}^{\infty} g_l S_l, \tag{3.131}$$

where g_l is determined by the boundary condition that $h_\phi B_\phi$ be constant on the boundary (so that $B_\mu(\mu_0) = 0$, as will be discussed below), and h_ϕ is a metric coefficient defined below. Tsuji then *numerically* showed that if $h_\phi B_\phi$ is set equal to S_l, the force-free relation is satisfied for $0 \le l \le 14$. The other components of $\boldsymbol{B}$ are derivable from $h_\phi B_\phi$ and satisfy the divergence and force-free conditions independent of a truncation in the summation in Eq. (3.131).

Buck pointed out in his 1965 paper that the difficulty in obtaining analytic solutions for the interior region of the toroid is due to the fact that the boundary conditions create a non-separable problem. This is intuitively clear since all solutions to the force-free equations are solutions to the Helmholtz equation, and one cannot expect to find solutions to the force-free equations by the method of separation of variables in coordinate systems where the Helmholtz equation does not separate. The Helmholtz equation only separates in coordinates such that the coordinate surfaces form a triply orthogonal system of confocal quadrics.[26] Unfortunately, toroidal coordinates belong to the class of coordinates associated with confocal cyclides. While the Laplace equation will separate in these coordinates, the Helmholtz equation will not.

Tsuji's work is based on that of Weston[27] who found that in cylindrical and rotational coordinate systems (see Appendix 3) one variable can be separated out of the Helmholtz equation. This then leaves a second order partial differential equation in two variables. For a certain class of coordinate systems, including toroidal, this equation is reducible to a recurrence set of ordinary differential equations in one variable, which can be solved by ordinary methods. Weston[28] has also obtained solutions to the Helmholtz equation in toroidal coordinates for the exterior problem; i.e., solutions which are non-singular and continuous in the region exterior to the torus $\mu = 0$.

The following sections will introduce toroidal coordinates and notation; derive the non-constant α force-free equations, if for no other purpose than to illustrate their intractability; and, after specializing to constant α solutions, describe the work of Weston and Tsuji outlined above.

3.3.1 Toroidal Coordinates

The relation between rectangular and toroidal coordinates is given by[29]

$$x = \frac{a \sinh\mu \cos\phi}{\cosh\mu - \cos\eta}, \quad y = \frac{a \sinh\mu \sin\phi}{\cosh\mu - \cos\eta}, \quad z = \frac{a \sin\eta}{\cosh\mu - \cos\eta}. \tag{3.132}$$

As illustrated in Figs. 3.6 and 3.7, μ ranges from 0 to ∞, with $\mu = 0$ corresponding to the z-axis and $\mu = \infty$ to the circle in the x,y-plane, centered at the origin, with radius a. The surface $\mu = \mu_0$ is a torus having a circular cross section of radius $a \operatorname{csch}\mu_0$. The centers of such cross sections lie on a circle of radius $a \coth\mu_0$ centered on the origin. That is, $\mu = \mu_0$ defines the torus

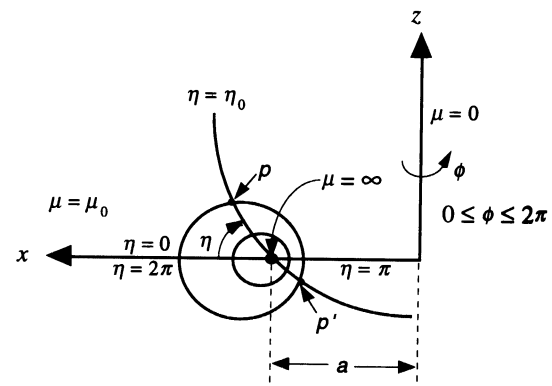

Fig. 3.6. Toroidal coordinates. Surfaces of constant μ are tori of circular cross section; $\mu = 0$ corresponds to the z-axis while $\mu = \infty$ to a circle of radius a, centered at the origin, and lying in the x,y-plane. The angular coordinate η ranges from 0 to 2π with $\eta = \pi$ corresponding to the disk bounded by $\mu = \infty$.

$$z^2 + (\rho - a \coth\mu_0)^2 = a^2 \operatorname{csch}^2\mu_0,$$

where $\rho = (x^2 + y^2)^{1/2}$, and $\eta = \eta_0$ defines the sphere

$$(z - a \cot\eta_0)^2 + \rho^2 = a^2 \csc^2\eta_0.$$

The geometric relations of Fig. 3.7 are readily derivable from elementary geometry and Eqs. (3.132).

Consider Fig. 3.6. The points p and p' have the same μ and ϕ coordinates ($\phi = 0$ corresponds to the positive x half of the x,z-plane) and, if these points are not to have the same η coordinate, the part of the sphere corresponding to the η coordinate above the x,y-plane must have a different value from the part below the plane. That is, in Fig. 3.6 $\eta = \eta_0$, $\eta_0 < \pi$ corresponds to a sphere of radius $a \csc\eta_0$ whose center is located on the z-axis at $z = a \cot\eta_0$. If the coordinate system is to be single valued, the part of the sphere above the x,y-plane and the part below must correspond to different values of η. How this is done is unimportant for what follows.

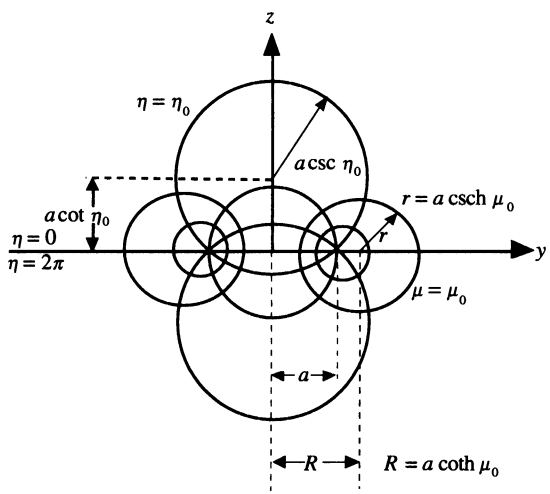

Fig. 3.7. Toroidal coordinates. $\eta = \eta_0$, for $\eta_0 < \pi$, corresponds to a sphere of radius $a\csc\eta_0$, whose center is located on the z-axis at $a\cot\eta_0$. For the coordinates to be single valued, the part of the sphere lying below the x,y-plane must have a different value than the part that lies above the plane. Note that $a^2 = R^2 - r^2$.

The metric coefficients for the coordinates defined by Eqs. (3.132) are

$$h_\mu = h_\eta = \frac{a}{\cosh\mu - \cos\eta}, \qquad h_\phi = \frac{a \sinh\mu}{\cosh\mu - \cos\eta}. \tag{3.133}$$

Recall from Section 3.2.1 that the h_i, $i = \mu,\eta,\phi$, are defined as $h_i = \sqrt{g_{ii}}$ where the metric tensor g_{ij} is the inner product of the basis vectors associated with a given coordinate system u^i. [The basis vectors $\boldsymbol{a}_i$ are given by $\boldsymbol{a}_i = \partial_{u^i}\boldsymbol{r}$] The metric coefficients allow the gradient, divergence, and curl to be written as:

$$\nabla\Phi = \frac{\hat{\mu}}{h_\mu}\partial_\mu\Phi + \frac{\hat{\eta}}{h_\eta}\partial_\eta\Phi + \frac{\hat{\phi}}{h_\phi}\partial_\phi\Phi, \tag{3.134}$$

$$\nabla\cdot\boldsymbol{B} = \frac{1}{h_\mu h_\eta h_\phi}\left[\partial_\mu(h_\eta h_\phi B_\mu) + \partial_\eta(h_\phi h_\mu B_\eta) + \partial_\phi(h_\mu h_\eta B_\phi)\right], \tag{3.135}$$

$$\nabla\times\boldsymbol{B} = \frac{1}{h_\mu h_\eta h_\phi}\begin{vmatrix} h_\mu\hat{\mu} & h_\eta\hat{\eta} & h_\phi\hat{\phi} \\ \partial_\mu & \partial_\eta & \partial_\phi \\ h_\mu B_\mu & h_\eta B_\eta & h_\phi B_\phi \end{vmatrix}. \tag{3.136}$$

3.3.2 Non-Constant α Field Equations

Equations (3.136) and (3.133) can be used to explicitly write the force-free relation $\nabla\times\boldsymbol{B} = \alpha\boldsymbol{B}$ in toroidal coordinates. The μ and η components of the relation yield

$$h_\eta B_\eta = -\frac{1}{\alpha h_\phi}\partial_\mu(h_\phi B_\phi) \tag{3.137}$$

and

$$h_\mu B_\mu = \frac{1}{\alpha h_\phi}\partial_\eta(h_\phi B_\phi). \tag{3.138}$$

Note that if $B_\mu(\mu_0) = 0$ is to be imposed as a boundary condition, $h_\phi B_\phi$ must be constant on the surface $\mu = \mu_0$. The ϕ component of the force-free relation results in

$$\partial_\mu(h_\eta B_\eta) - \partial_\eta(h_\mu B_\mu) = \frac{\alpha h_\eta}{\sinh\mu}(h_\phi B_\phi). \tag{3.139}$$

It is clear that these equations may be combined by substituting for $h_\eta B_\eta$ and $h_\mu B_\mu$ in Eq. (3.139) from Eqs. (3.137) and (3.138). Before doing so, however, consider Eq. (3.135). Specializing to axial symmetry and requiring the divergence of $\boldsymbol{B}$ to vanish results in

$$\partial_\mu(h_\eta h_\phi B_\mu) + \partial_\eta(h_\phi h_\mu B_\eta) = 0. \tag{3.140}$$

Substituting for B_μ and B_η from Eqs. (3.137) and (3.138) yields

$$(\partial_\eta\alpha)\,\partial_\mu(h_\phi B_\phi) - (\partial_\mu\alpha)\,\partial_\eta(h_\phi B_\phi) = 0. \tag{3.141}$$

This means that α is a function of only $h_\phi B_\phi$; i.e., $\alpha = \alpha(h_\phi B_\phi)$. One additional relation is needed: as was seen in earlier chapters, the force-free relation implies that $\nabla\alpha\cdot\boldsymbol{B} = 0$; this means, using Eq. (3.134), that since α is not a function of ϕ by symmetry,

$$\partial_\eta\alpha = -\frac{H_\mu}{H_\eta}\partial_\mu\alpha. \tag{3.142}$$

Now let $\Psi = h_\phi B_\phi$. Combining Eqs. (3.137), (3.138), (3.139) and (3.142) then gives

$$\partial_\mu\left[\frac{1}{h_\phi}\partial_\mu\Psi\right] + \partial_\eta\left[\frac{1}{h_\phi}\partial_\eta\Psi\right] + \frac{\partial_\mu\alpha}{\alpha h_\phi}\left[\frac{B_\mu}{B_\eta}\partial_\eta\Psi - \partial_\mu\Psi\right] + \frac{\alpha^2 h_\eta}{\sinh\mu}\Psi = 0. \tag{3.143}$$

Equations (3.137) and (3.138) can again be used to substitute for B_μ and B_η in this expression. Also, let $'$ designate differentiation with respect to Ψ. Equation (3.143) can then be written as

$$\alpha' - h_\phi\frac{\partial_\mu\left[\frac{1}{h_\phi}\partial_\mu\Psi\right] + \partial_\eta\left[\frac{1}{h_\phi}\partial_\eta\Psi\right]}{\left[(\partial_\eta\Psi)^2 + (\partial_\mu\Psi)^2\right]}\alpha - \frac{h_\eta{}^2\Psi}{\left[(\partial_\eta\Psi)^2 + (\partial_\mu\Psi)^2\right]}\alpha^3 = 0. \tag{3.144}$$

If the coefficients of α in this expression were functions of Ψ alone, this would be Bernoulli's equation and it would be readily soluble for α as a function of Ψ. Unfortunately, this is not the case, and the author is unaware of any technique that would yield an exact solution. The next section therefore restricts the problem to the case where α is constant.

3.3.3 Solution in Toroidal Coordinates for Constant α

If α is constant, Eq. (3.143) can be written as

$$\partial_\mu\left[\frac{1}{h_\phi}\partial_\mu\Psi\right] + \partial_\eta\left[\frac{1}{h_\phi}\partial_\eta\Psi\right] + \alpha^2\frac{h_\eta h_\mu}{h_\phi}\Psi = 0. \tag{3.145}$$

Solving this equation solves the problem since B_ϕ is given by Ψ/h_ϕ and Eqs. (3.137) and (3.138) written in terms of Ψ yield the other components of the field; i.e.,

$$B_\eta = -\frac{1}{\alpha h_\eta h_\phi}\partial_\mu\Psi \tag{3.146}$$

and

$$B_\mu = \frac{1}{\alpha h_\mu h_\phi}\partial_\eta\Psi. \tag{3.147}$$

Solutions to Eq. (3.145) may be obtained by use of the technique developed in Section 3.1.2. There it was found that if Ψ satisfied the Helmholtz equation, the field given by Eq. (3.26) (where $\boldsymbol{B}$ was written in terms of Ψ and a constant vector), would satisfy the constant α force-free equations.

For toroidal coordinates, the obvious constant vector to use in Eq. (3.26) is $\hat{z} = \nabla z$. The third of Eqs. (3.132) for z, and Eq. (3.134) for the gradient operation in toroidal coordinates, give as a representation of $\hat{z}$ in these coordinates

$$\nabla\chi = -\frac{\sin\eta\sinh\mu}{\cosh\mu - \cos\eta}\hat{\mu} + \frac{\cosh\mu\cos\eta - 1}{\cosh\mu - \cos\eta}\hat{\eta}, \tag{3.148}$$

where χ represents z in toroidal coordinates; i.e.,

$$\chi = \frac{a\sin\eta}{\cosh\mu - \cos\eta}. \tag{3.149}$$

The expression for $\hat{z}$ in toroidal coordinates can also be obtained by the process of writing $\boldsymbol{r} = x\hat{\boldsymbol{i}} + y\hat{\boldsymbol{j}} + z\hat{\boldsymbol{k}}$ and substituting from Eq. (3.132) for x, y, and z. The choice of ϕ is arbitrary, and for simplicity can be chosen to be $\phi = 0$. The unit vectors orthogonal to the coordinate surfaces are

$$\hat{\mu} = \frac{\frac{\partial\boldsymbol{r}}{\partial\mu}}{\left|\frac{\partial\boldsymbol{r}}{\partial\mu}\right|}, \qquad \hat{\eta} = \frac{\frac{\partial\boldsymbol{r}}{\partial\eta}}{\left|\frac{\partial\boldsymbol{r}}{\partial\eta}\right|}. \tag{3.150}$$

These lead to two equations expressing $\hat{\mu}$ and $\hat{\eta}$ in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{k}}$. Eliminating $\hat{\boldsymbol{i}}$, one obtains for $\hat{\boldsymbol{k}}$ Eq. (3.148). Note that since toroidal coordinates are orthogonal, the unit vectors tangent to the coordinate curves given by Eq. (3.150) coincide with the basis vectors formed from the normal to the coordinate surfaces determined by the gradient.

Equation (3.26) for the field may now be written as

$$\boldsymbol{B} = \alpha(\nabla\Psi\times\nabla\chi) + \nabla\times(\nabla\Psi\times\nabla\chi) \tag{3.151}$$

or, equivalently,

$$\boldsymbol{B} = \alpha(\nabla\times\Psi\nabla\chi) + \nabla\times(\nabla\times\Psi\nabla\chi), \tag{3.152}$$

where $\nabla\chi$ represents the unit vector in the z-direction given by Eq. (3.148). If either of Eqs. (3.151) or (3.152) is substituted into the force-free relation, the result is the Helmholtz equation, as in Section 3.1.2.

Weston found that if the solution to the Helmholtz equation $\nabla^2\Psi + \alpha^2\Psi = 0$ is expressed in terms of a power series in $(s - \cos\eta)^{-1}$, where $s = \cosh\mu$, the differential equations for the coefficients of the series take on a simpler form. For the exterior problem, there is an even and an odd solution:

$$\Psi_e(s,\eta,\phi) = e^{im\phi}\sum_{r=T}^{\infty} A_r(s)(s-\cos\eta)^{-r} \qquad \text{(even)} \tag{3.153}$$

and

$$\Psi_o(s,\eta,\phi) = e^{im\phi}\sin\eta\sum_{r=T'}^{\infty} B_r(s)(s-\cos\eta)^{-r}, \qquad \text{(odd)} \tag{3.154}$$

where $m = 0, \pm1, \pm2, \ldots$ so that the solutions are periodic in the angle ϕ. The coefficients $A_r(s)$ and $B_r(s)$ satisfy the ordinary differential equations

$$\begin{gathered}(s^2-1)A_r'' + 2sA_r' - A_r\left[r(r+1) + \frac{m^2}{s^2-1}\right] + \\ \left\{\alpha^2a^2A_{r-2} - (2r-1)\left[(s^2-1)A_{r-1}' - s(r-1)A_{r-1}\right]\right\} = 0\end{gathered} \tag{3.155}$$

and

$$\begin{gathered}(s^2-1)B_r'' + 2sB_r' - B_r\left[r(r-1) + \frac{m^2}{s^2-1}\right] + \\ \left\{\alpha^2a^2B_{r-2} - (2r-1)\left[(s^2-1)B_{r-1}' - s(r-2)B_{r-1}\right]\right\} = 0,\end{gathered} \tag{3.156}$$

where the prime denotes differentiation with respect to s.

Since A_r and B_r vanish for $r < T$ and $r < T'$ respectively, the expressions in curly brackets in Eqs. (3.155) and (3.156) will vanish for $r = T$ and $r = T'$ respectively. This leaves

$$(s^2-1)A_T'' + 2sA_T' - A_T\left[T(T+1) + \frac{m^2}{s^2-1}\right] = 0 \tag{3.157}$$

and

$$(s^2-1)B_{T'}'' + 2sB_{T'}' - B_{T'}\left[T'(T'-1) + \frac{m^2}{s^2-1}\right] = 0. \tag{3.158}$$

Equation (3.157), for $m \le T$, is the differential equation for the associated Legendre functions $P_T^m(s)$ and $Q_T^m(s)$. The solutions to Eq. (3.158) are then given by $P_{T'-1}^m(s)$ and $Q_{T'-1}^m(s)$.

Let $\omega_r^m(s)$ denote either $P_r^m(s)$ or $Q_r^m(s)$. Given the solution to the homogeneous equations (3.157) and (3.158), the solution to the inhomogeneous equations (3.155) and (3.156) can be written as

$$A_r(s) = \omega_r^m(s)\int_{c_1}^{s}\frac{dz}{(1-z^2)[\omega_r^m(z)]^2}\int_{c_2}^{z}F_r(x)\omega_r^m(x)\,dx \tag{3.159}$$

and

$$B_r(s) = \omega_{r-1}^m(s)\int_{c_1'}^{s}\frac{dz}{(1-z^2)[\omega_{r-1}^m(z)]^2}\int_{c_2'}^{z}G_r(x)\omega_{r-1}^m(x)\,dx, \tag{3.160}$$

where the constants c_1, c_2, c_1', and c_2' are determined from the boundary conditions, and $F_r(s)$ and $G_r(s)$ correspond to the inhomogeneous parts of Eqs. (3.155) and (3.166). That is,

$$F_r(s) = \alpha^2a^2A_{r-2} - (2r-1)\left[(s^2-1)A_{r-1}' - s(r-1)A_{r-1}\right] \tag{3.161}$$

and

$$G_r(s) = \alpha^2a^2B_{r-2} - (2r-1)\left[(s^2-1)B_{r-1}' - s(r-2)B_{r-1}\right]. \tag{3.162}$$

For the exterior problem, which includes the region $s_0 \ge s \ge 1$ outside the torus $s = s_0$, solutions involving the associated Legendre functions $Q_T^m(s)$ are singular on the surface $s = 1$ corresponding to the z-axis. Thus, non-singular solutions will only involve $P_T^m(s)$.

To obtain explicit expressions for the solutions to the Helmholtz equation, Weston defined a basic set of solutions $\psi_{eT}^m(P)$ and $\psi_{oT'}^m(P)$ satisfying Eqs. (3.153) and (3.154) respectively. The coefficients $A_r(s)$ and $B_r(s)$ are given by Eqs. (3.159) and (3.160) with $c_1 = c_2 = c_1' = c_2' = 1$, and $A_T(s)$ and $B_{T'}(s)$ equal to $P_T^{-|m|}(s)$ and $P_{T'-1}^{-|m|}(s)$ respectively. The negative superscript allows the case $|T| < |m|$ and $|T'-1| < |m|$, where the associated Legendre functions $P_T^{|m|}(s)$ and $P_{T'-1}^{|m|}(s)$ do not exist, to be included. By a process of mathematical induction, it can be shown that any solution of the form $\{\psi_{eT}^m(P) + \psi_{oT'}^m(P)\}$, with *arbitrary* constants of integration, can be expressed as a linear combination of the explicit solutions $\psi_{ep}^m(P)$ and $\psi_{op'}^m(P)$, where $p = T, T+1, \ldots$ and $p' = T', T'+1, \ldots$.

Choosing the constants of integration in Eqs. (3.159) and (3.160) to be equal to the fixed value of unity then allows the expressions for $A_r(s)$ and $B_r(s)$ to be integrated, and ultimately results in the following explicit solutions for $\psi_{eT}^m(P)$ and $\psi_{oT'}^m(P)$:

$$\psi_{eT}^m(P) = e^{im\phi}\sum_{r=T}^{\infty}(s-\cos\eta)^{-r}\sum_{t=0}^{[(r-T)/2]}(\alpha a)^{2t}A_r^t(s^2-1)^{(r-T-t)/2}P_{-T-t-1}^{-M-r+T+t}(s) \tag{3.163}$$

$$\psi^m_{oT'}(P) = e^{im\phi}\sin\eta\sum_{r=T'}^{\infty}(s-\cos\eta)^{-r}\sum_{t=0}^{[(r-T')/2]}(\alpha a)^{2t}B^t_r(s^2-1)^{(r-T'-t)/2}P^{-M-r+T'+t}_{-T'-t}(s), \quad (3.164)$$

where

$$A^t_r = \frac{(-1)^{t+r-T}\,\Gamma(-T-t+1/2)\,\Gamma(-2T+r-2t+M)}{2^t\,(t)!\,(r-T-2t)!\,\Gamma(-r+t+1/2)\,\Gamma(M-T)} \quad (3.165)$$

and

$$B^t_r = \frac{(-1)^{t+r-T'}\,\Gamma(-T'-t+1/2)\,\Gamma(-2T'+r-2t+M+1)}{2^t\,(t)!\,(r-T'-2t)!\,\Gamma(-r+t+1/2)\,\Gamma(M-T'+1)}. \quad (3.166)$$

Here, $M = |m|$ and the notation $[x]$ in the upper limit of the summation means that integer such that $x-1 < [x] \le x$; e.g., for $r = T+3$, $[(r-T)/2] = 1$.

The even solutions of η, $\psi^m_{eT}(P)$, are uniformly convergent in the region $1 \le s \le \infty$ and $0 \le \eta \le 2\pi$, and are therefore continuous in that region. However, the partial derivatives with respect to η of the even solutions are discontinuous at $\eta = 0$ (or $\eta = 2\pi$), except when $M - T$ is zero or a negative integer. The odd solutions, $\psi^m_{oT'}(P)$, are discontinuous at $\eta = 0$ unless $M - T' + 1$ is zero or a negative integer. The conditions on $M - T$ and $M - T' + 1$ derive from the fact that the Gamma function $\Gamma(z)$ is analytic over the finite part of the complex plane with the exception of the points $z = 0, -1, -2, \ldots$ where it has simple poles. While the partial derivatives with respect to η, as well as the odd solutions themselves, are discontinuous at $\eta = 0$, there do exist linear combinations of these solutions that are continuous at $\eta = 0$.

Tsuji, in approaching the interior problem for which $\mu \ge \mu_0$ or $s \ge s_0$, replaced the associated Legendre functions of the first kind, $P^m_T(s)$, in Eqs. (3.163) and (3.164) with $Q^m_T(s)$, the associated Legendre functions of the second kind. However, Weston used the relation $P^{\mu}_{-\nu-1}(z) = P^{\mu}_{\nu}(z)$ in his derivation of explicit expressions for $\psi^m_{eT}(P)$ and $\psi^m_{oT'}(P)$. This relation is not generally valid for $Q^{\mu}_{\nu}(z)$, for which one has[30]

$$Q^{\mu}_{-\nu-1}(z) = \frac{\left\{-\pi e^{i\mu\pi}\cos\nu\pi\, P^{\mu}_{\nu}(z) + Q^{\mu}_{\nu}(z)\sin[\pi(\nu+\mu)]\right\}}{\sin[\pi(\nu-\mu)]}.$$

If ν is restricted to half an odd integer the term involving $\cos\nu\pi$ will vanish and $\sin[\pi(\nu+\mu)]$ will be ± 1 and have the same sign as $\sin[\pi(\nu-\mu)]$ so that $Q^{\mu}_{-\nu-1}(z) = Q^{\mu}_{\nu}(z)$. Now in Eq. (3.163) the lower index on Q substituted for P is $\{-T-t-1\}$ with the summation over t starting at zero. Consider $T = p - 1/2$, where p is an integer. If $\{-p+1/2-t-1\}$ is identified with $\{-\nu-1\}$, then $\nu = p+t-1/2$, or $(2p+2t-1)/2$. This is always half an odd integer.

Restricting T to $p - 1/2$ in the lower index of Q and then changing the range of the summation over r to be $0 \le r \le \infty$ (i.e., let $r \to r + T$) in Eq. (3.163) gives

$$\psi^m_{ep}(P) = e^{im\phi}\sum_{r=0}^{\infty}(s-\cos\eta)^{-r-p+1/2}\sum_{t=0}^{[r/2]}(\alpha a)^{2t}A^t_r(s^2-1)^{(r-t)/2}P^{-M-r+t}_{-p-t-1/2}(s), \quad (3.167)$$

where the role of T in ψ^m_{eT} has been taken by p. A^t_r is now given by

$$\begin{aligned} A^t_r &= \frac{(-1)^{t+r}\,\Gamma(-p+1-t)\,\Gamma(r-p+1/2-2t+M)}{2^t\,(t)!\,(r-2t)!\,\Gamma(-r-p+1+t)\,\Gamma(M-p+1/2)} \\ &= \frac{(-1)^{t+r}\,(t-r-p+1)_{r-2t}\,\Gamma(M-p+r-2t+1/2)}{2^t\,(t)!\,(r-2t)!\,\Gamma(M-p+1/2)}, \end{aligned} \quad (3.168)$$

where the notation $(a)_n = \Gamma(a+n)/\Gamma(a)$ has been used.

Similarly, for the odd function ψ^m_{op}, by letting $r \to r + T'$ and setting $T' = p + 1/2$, one obtains from Eqs. (3.164) and (3.166)

$$\psi^m_{op}(P) = e^{im\phi}\sin\eta\sum_{r=0}^{\infty}(s-\cos\eta)^{-r-p-1/2}\sum_{t=0}^{[r/2]}(\alpha a)^{2t}B^t_r(s^2-1)^{(r-t)/2}P^{-M-r+t}_{-p-t-1/2}(s) \quad (3.169)$$

and

$$\begin{aligned} B^t_r &= \frac{(-1)^{t+r}\,\Gamma(-p-t)\,\Gamma(r-p-2t+M+1/2)}{2^t\,(t)!\,(r-2t)!\,\Gamma(-r-p+t)\,\Gamma(M-p+1/2)} \\ &= \frac{(-1)^{t+r}\,(t-r-p)_{r-2t}\,\Gamma(M-p+r-2t+1/2)}{2^t\,(t)!\,(r-2t)!\,\Gamma(M-p+1/2)}. \end{aligned} \quad (3.170)$$

Unfortunately, the infinite series in Eqs. (3.167) and (3.169) do not converge. To obtain convergence, Tsuji considered a linear combination of the solutions given by these equations. He used

$$U_{j,m} = \frac{1}{\sqrt{2\pi}}\sum_{p=-m}^{\infty}\frac{\Gamma(M-p+1/2)}{(m+p)!}\left(\frac{\alpha^2 a^2}{2}\right)^p \psi^m_{jp}\,e^{-im\phi}, \quad (3.171)$$

where $j = (e, o)$ stands for the even and odd solution respectively. It can now be shown that the summation converges, so that $U_{j,m}e^{im\phi}$ is an analytic solution to the Helmholtz equation in toroidal coordinates for the interior problem.

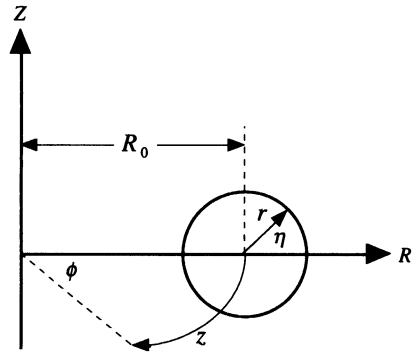

Fig. 3.8. Toroidally curved cylindrical coordinates r, η, z and their relation to the conventional cylindrical coordinates R, ϕ, Z.

In his first paper, Tsuji used only the even function ψ^1_{ep} for $m = 1$ and formed a linear combination of the functions U that satisfied the requirement that the solutions formed by the linear combination approach those for the cylindrical configuration when the aspect ratio $R/r \to \infty$. As mentioned in the beginning of this section, he then showed, by numerical calculation, that these solutions satisfy the force-free relation. Before forming the linear combination under discussion, the form that the solution must take in the limit that $R/r \to \infty$ will be derived.

Toroidally curved cylindrical coordinates are shown in Fig. 3.8. These coordinates are related to the usual cylindrical coordinates $\{R, \phi, Z\}$ by

$$R = R_0 + r\cos\eta, \qquad Z = r\sin\eta, \qquad \phi = -\frac{z}{R_0}. \tag{3.172}$$

In the usual cylindrical coordinates, if $\boldsymbol{B}$ is assumed to have the form

$$\boldsymbol{B} = \alpha(\nabla\times\Phi\hat{\phi}) + \nabla\times(\nabla\times\Phi\hat{\phi}) \tag{3.173}$$

and substituted into the force-free relation, Φ will satisfy the Helmholtz equation. In cylindrical coordinates this takes the form

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial Z^2} + \alpha^2\Phi = 0, \tag{3.174}$$

or

$$\frac{\partial^2\Phi}{\partial R^2} + \frac{\partial^2\Phi}{\partial Z^2} + \frac{1}{R}\frac{\partial\Phi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\Phi}{\partial\phi^2} + \alpha^2\Phi = 0. \tag{3.175}$$

Assume that the ϕ dependence is $e^{in\phi}$, and let $\Phi = \psi/R^{1/2}$. Then Eq. (3.175) becomes

$$\frac{\partial^2\psi}{\partial R^2} + \frac{\partial^2\psi}{\partial Z^2} + \left(\alpha^2 - \frac{n^2 - 1/4}{R^2}\right)\psi = 0. \tag{3.176}$$

Observe that the first two terms of Eq. (3.176) constitute the 2-dimensional Laplacian in Cartesian coordinates. The corresponding toroidally curved cylindrical coordinates of Eq. (3.172) are essentially polar coordinates in the plane. Thus, Eq. (3.176) is readily expressed in terms of the curved cylindrical coordinates.[31] The result is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\eta^2} + \left(\alpha^2 - \frac{n^2 - 1/4}{R^2}\right)\psi = 0. \tag{3.177}$$

Now if $r/R_0 << 1$, from the first of Eqs. (3.172) it is clear that R^2 in Eq. (3.177) may be replaced with R_0^2, and if R_0^2 is large compared to $n^2 - 1/4$, this term may be neglected. The resulting equation may be separated, in terms of a separation constant m^2, by writing $\psi = g(\eta)h(r)$ to yield two equations:

$$\begin{aligned} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + \left(\alpha^2 - \frac{m^2}{r^2}\right)h &= 0 \\ \frac{\partial^2 g}{\partial\eta^2} + m^2 g &= 0 \end{aligned} \tag{3.178}$$

The first of these is Bessel's equation, and both are readily solved to obtain

$$\psi = J_m(\alpha r)e^{im\eta}. \tag{3.179}$$

The linear combination $S_{j,m}$ of the functions $U_{j,m}$ must then satisfy

$$\begin{aligned} S_{e,m} &\to J_m(\alpha r)\cos m\eta \\ S_{o,m} &\to J_m(\alpha r)\sin m\eta. \end{aligned} \tag{3.180}$$

Tsuji has shown that the following linear combinations of the $U_{j,m}$ of Eq. (3.171) will satisfy the requirements of Eqs. (3.180):

$$S_{e,m} = (-\alpha a)^m \sum_{k=0}^{m} \left\{ \sum_{\nu=[(k+1)/2]}^{k} \frac{\lambda_{e,\nu}^{m,k}}{(\alpha a)^{2\nu}} \right\} U_{e,m-k} \tag{3.181}$$

$$S_{o,m} = -(-\alpha a)^m \sum_{k=0}^{m-1} \left\{ \sum_{\nu=[(k+1)/2]}^{k} \frac{\lambda_{o,\nu}^{m,k}}{(\alpha a)^{2\nu}} \right\} U_{o,m-k}. \tag{3.182}$$

While numerical values for the $\lambda_{j,\nu}^{m,k}$ have been calculated by Tsuji, these coefficients have not been expressed as a function of m, k, and ν.

The magnetic field for the interior region of the torus $s \geq s_0$ (or $\mu \geq \mu_0$) can now be obtained by substituting $\Psi = S_j e^{im\phi}$ into Eq. (3.152), where

$$S_j = \sum_{n=\delta_{j,o}}^{\infty} a_{j,n} S_{j,n}; \qquad \delta_{j,o} = \begin{cases} 0 & j = e \\ 1 & j = o \end{cases} \tag{3.183}$$

and the $a_{j,n}$ are constants determined from the boundary conditions.

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Chapter 4

FIELD TOPOLOGY

The equation $\nabla\alpha\cdot\boldsymbol{B} = 0$ introduced in Chapter 1 can be written in component form as

$$B^i\frac{\partial\alpha}{\partial x^i} = 0. \tag{4.1}$$

In the theory of linear partial differential equations of the first order, the vector $\boldsymbol{B}$ is known as the characteristic vector and its direction as the characteristic direction. The condition imposed by Eq. (4.1) on the dependent variable $\alpha(x^i)$ is independent of the magnitude of $\boldsymbol{B}$. Since the functions $B^i(x^j)$ transform as the components of a contravarient vector, the characteristic direction is independent of any coordinate transformation. The infinitesimal displacement $\mathrm{d}x^i = B^i\mathrm{d}t$ is therefore well defined at each point.

Suppose that $\alpha(x^i)$ satisfies Eq. (4.1). Then under the displacement $\mathrm{d}x^i = B^i\mathrm{d}t$,

$$\mathrm{d}\alpha(x^i) = \frac{\partial\alpha}{\partial x^i}\mathrm{d}x^i = \frac{\partial\alpha}{\partial x^i}B^i\mathrm{d}t = 0. \tag{4.2}$$

This means that the displacement $\mathrm{d}x^i = B^i\mathrm{d}t$ is tangent to the level or integral surface $\alpha(x^i)$ = constant. The characteristic direction is tangent to every integral surface.

Members of the family of curves whose tangent at each point is the characteristic direction are known as characteristic curves, and satisfy the system of ordinary differential equations

$$\frac{\mathrm{d}x^i}{\mathrm{d}t} = B^i. \tag{4.3}$$

Note that t now appears as an independent variable, and Eq. (4.3) has a family of solutions

of the form $x^i(t)$ which represent integral curves of the characteristic vector field $\boldsymbol{B}$. The choice of the parameter t is arbitrary since using any other parameter along the curve is equivalent to replacing the B^i by proportional quantities.

Equation (4.2) shows that $\alpha(x^i)$ is constant along any characteristic or integral curve. Thus, if an integral surface contains one point of an integral curve, it must contain the entire curve. Since an integral curve passes through each point of an integral surface, each of these curves must lie entirely within the integral surface. An integral surface may consequently be thought of as being constructed from integral curves. The integral curves correspond to the field lines of the magnetic field $\boldsymbol{B}$, and the integral surfaces, when they exist, are the magnetic surfaces. $\alpha(x^i)$ is then a constant on the magnetic surfaces.

Since t does not explicitly appear as a variable of the functions $B^i(x^j)$, the system of equations given by Eq. (4.3) is autonomous and t may be eliminated to yield a second system of equations

$$\frac{dx^1}{B^1} = \frac{dx^2}{B^2} = \frac{dx^3}{B^3} \tag{4.4}$$

which define a set of trajectories in 3-dimensional space. The integrals of this system, given by

$$F_i(x^j) = c_i, \qquad i = 1, 2 \tag{4.5}$$

correspond to two surfaces that depend on the independent parameters c_1 and c_2. The trajectories defined by Eq. (4.4) are the intersections of these surfaces, and correspond to the field lines of the magnetic field with components B^1, B^2, and B^3.

The components of the gradient in Eq. (4.1) transform as a covariant vector field and geometrically correspond to a field of surface elements. An integral surface $\alpha(x^i)$ = constant is a surface which envelopes this field of surface elements. Since a field of surface elements does not necessarily possess an envelope, an integral surface may not exist.

There are two total differential equations which are adjoint to the partial differential equation given by Eq. (4.1). These are found by determining the two covariant vectors with components $B_i^{(k)}$, k = 1,2, such that $B_i^{(k)}B^i = 0$. The adjoint system is then

$$B_i^{(k)}dx^i = 0, \qquad k = 1, 2. \tag{4.6}$$

Since $B_i^{(k)}B^i = 0$, dx^i may be written as $dx^i = B^i dt$, which is the same as Eq. (4.3) for the characteristics of Eq. (4.1).

An integral of Eq. (4.6) is defined to be a function $\alpha(x^i)$ such that

$$d\alpha = \frac{\partial\alpha}{\partial x^i}dx^i = 0 \tag{4.7}$$

when dx^i satisfies Eq. (4.6). Thus, $\partial\alpha/\partial x^i$ is a multiple of $B_i^{(k)}$ and the two quantities define the same surface element which, at any point, is tangent to the integral surface $\alpha(x^i)$ = constant. If this surface (which is the envelope of the field of surface elements defined by $B_i^{(k)}$ or $\partial\alpha/\partial x^i$) exists, $B_i^{(k)}dx^i$ will have an integrating factor $\mu^{(k)}(x^i)$ such that

$$\mu^{(k)}B_i^{(k)}dx^i = d\alpha \tag{4.8}$$

and Eq. (4.1) will then have two independent solutions for $\alpha(x^i)$ corresponding to $k = 1, 2$, as mentioned in Chapter 1. Note that the converse is not true. An exact differential equation has an integral which exists locally, but its global existence depends on the topology of the region.

4.1 Magnetic Surfaces

Magnetic fields do not, in general, form magnetic surfaces. Such surfaces arise in magnetohydrostatic equilibria and for some highly symmetric field configurations. In the case of the latter, Eq. (4.4) for the lines of force will admit an exact integral which is the equation for the magnetic surface.

If it is assumed that the field is not singular -- that it does not have a zero at any point of the surface -- it can be shown that the surface cannot be simply connected. Cowling's theorem for force-free fields, discussed in Section 4.1.1, is a special case of this observation.

First, some definitions: The index of a closed curve is defined as the number of revolutions made by a vector field in traversing the curve once in the counter clockwise (positive) direction. Figure 4.1 gives the example of a saddle point with index equal to −1. The index of any (and hence of every) sufficiently small positively oriented circle centered on an isolated singular point of an arbitrary vector field is called the index of the singular point.

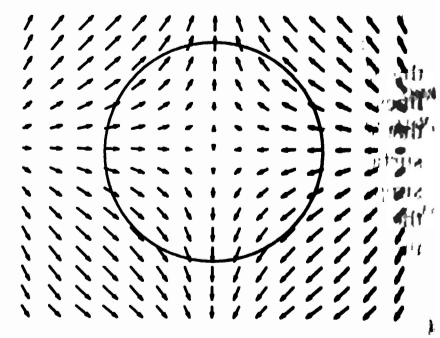

Figure 4.1. A Vector field with index −1. As the circle is traversed in the positive direction, the field rotates once in the negative direction.

Partitioning a surface into curvilinear triangles that are edgewise adjacent to each other is called a triangulation of the surface. For any triangulated surface, the number $\chi = v - e + f$, where v is the number of vertices, e the number of edges, and f the number of faces is called the Euler characteristic.

Now the Poincaré-Hopf theorem[1] states that the sum of the indices of the singular points of a vector field on any compact 2-dimensional manifold is independent of the field, and is given by the Euler characteristic χ. In addition, any closed, orientable surface is

homeomorphic (topologically equivalent) to a sphere with p-handles,[2] which has the Euler characteristic $2 - 2p$. Thus, a simply connected surface corresponds to $p = 0$, and Euler characteristic equal to 2.

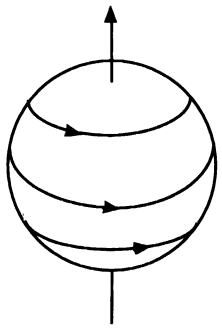

Figure 4.2. A vector field on a sphere. The Euler characteristic is the sum of the indices of the two singular points, each with index +1, at the poles.

Since the Poincaré-Hopf theorem allows a vector field to be chosen arbitrarily on a simply connected surface, consider the one shown in Figure 4.2. From Ampere's circuital law, it is clear that if the field on the sphere represents a magnetic field, the singular points correspond to a current entering at the south pole and leaving at the north pole of the sphere.

The fact that the Euler characteristic is equal to 2 means that a simply connected surface cannot have a non-singular vector field. For the field to be non-singular, the Euler characteristic must vanish. There is a theorem by Hopf[3] which states that the torus and the Klein bottle are the only smooth, compact, connected surfaces without boundary that can have a non-singular vector field. *Thus, a non-singular magnetic surface can be expected to have the topology of a torus.*[4]

4.1.1. Topology of Non-Constant α Solutions

For force-free magnetic fields with non-constant α, there is a theorem by Cowling[5] that shows that the surface cannot be simply connected. The proof is along the following lines: As mentioned in Chapter 1 and discussed in the introduction to this chapter, the fact that $\nabla\alpha\cdot\boldsymbol{B} = 0$ means that α is a constant on any field line, and when the field lines cover a surface, α is a constant on that surface. If the field is non-singular, and the surface is simply connected, the lines of force will form closed curves on the surface.[6] Let s denote the arc length along such a closed field line Γ, lying in the surface $\alpha =$ constant, and let Σ be the area bounded by Γ. Then, by Stokes' theorem,

$$\oint_\Gamma \boldsymbol{B}\cdot d\boldsymbol{s} = \int_\Sigma (\nabla\times\boldsymbol{B})\cdot d\boldsymbol{\Sigma} = \int_\Sigma \alpha\boldsymbol{B}\cdot d\boldsymbol{\Sigma} = \alpha\int_\Sigma \boldsymbol{B}\cdot d\boldsymbol{\Sigma} = 0, \tag{4.9}$$

where the force-free field equation $\nabla\times\boldsymbol{B} = \alpha\boldsymbol{B}$ has been used, and α may be removed from under the integral sign since it is constant on the surface. Thus, if the field is assumed to be non-zero, there is a contradiction since the first integral is positive definite and cannot vanish. Cowling concluded that the assumption that the surface $\alpha =$ constant is simply connected must be false.

Arnold[7] has shown that if $\boldsymbol{B}$ is a divergence-free field on a three dimensional closed orientable Riemannian manifold which satisfies $\boldsymbol{B}\times(\nabla\times\boldsymbol{B}) = \nabla\psi$, then the field lines of $\boldsymbol{B}$ and $\nabla\times\boldsymbol{B}$ lie on tori which are given by $\psi =$ constant. When $\nabla\psi = 0$ (the force-free case where $\nabla\times\boldsymbol{B} = \alpha\boldsymbol{B}$), the field lines will also lie on 2-dimensional tori, provided the field is non-singular and, α is not constant. If α is constant, the field can have more complicated topologies, which will be discussed in Section 4.1.2.

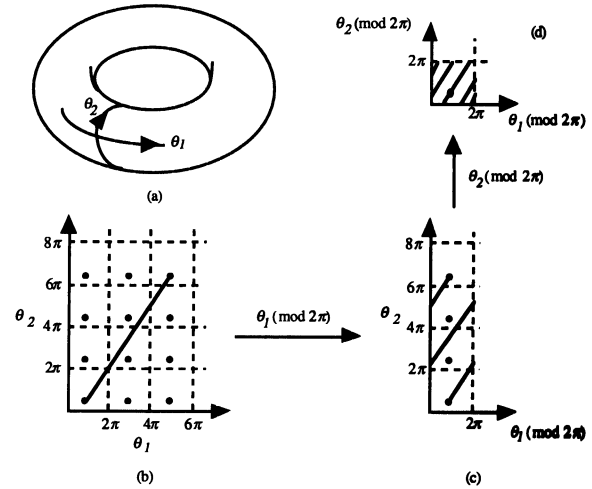

Figure 4.3. (a) angular coordinates on the 2-dimensional torus; (b) a closed field line on the torus for which ω_1/ω_2=2/3; (c) and (d) show the effect of imposing θ_1 (mod 2π) and θ_2 (mod 2π). The topology of the torus is obtained by identifying opposite sides of a rectangle.

Angular coordinates, $0 \le \theta_1 \le 2\pi$ and $0 \le \theta_2 \le 2\pi$, can be defined on a 2-dimensional torus such that any point is specified by θ_i (mod 2π), $i = 1,2$. As shown in Fig. 4.3(a), θ_1 can be pictured as going around the torus the "long way" and θ_2 the "short way." If the magnetic field lines constrained to the torus are such as to represent the trajectories of a uniform flow, parameterized by t, they will satisfy the set of equations

$$\dot\theta_1 = \omega_1, \qquad \dot\theta_2 = \omega_2, \tag{4.10}$$

The two numbers ω_1 and ω_2 are said to be rationally independent[8] if

$$a_1\,\omega_1 + a_2\,\omega_2 = 0 \tag{4.11}$$

implies that $a_1 = a_2 = 0$, where a_1 and a_2 are integers, and rationally dependent if Eq. (4.11) holds but $a_1^2 + a_2^2 \neq 0$.

Now if ω_1 and ω_2 are rationally dependent so that ω_1/ω_2 is a rational number, every field line on the torus satisfying Eq. (4.10) is closed. This is seen in the map of the torus shown in Fig. 4.3 for $\omega_1/\omega_2 = 2/3$. The figure also shows the effect of imposing θ_1 (mod 2π) and θ_2 (mod 2π). Notice that the topology of the torus is obtained by identifying opposite sides of a rectangle. If ω_1/ω_2 is a rational number, ω_1 and ω_2 may be chosen so that they do not have a common divisor other than unity, and are therefore relatively prime. For $\omega_1 \ge 2$ and $\omega_2 \ge 3$, the field line is knotted and corresponds to the torus knot[9] K_{ω_1,ω_2}.

On the other hand, if ω_1 and ω_2 are rationally independent then ω_1/ω_2 is an irrational number and a field line satisfying Eq. (4.10) is everywhere dense on the torus; i.e., the field line will come arbitrarily close to any point on the surface of the torus. The closed field line given by $\omega_1/\omega_2 = 2/3$ requires five lines for its representation by the map of the torus shown in Fig. 4.3(d). A field line that is dense on the torus would completely fill such a map.

4.1.2. Topology of Constant α Solutions

Hénon[10] has performed a numerical experiment to illustrate the complicated behavior of the field lines for a force-free field with constant α. He used the form of Eqs. (4.3) to study the system of equations:

$$\frac{dx}{dt} = B_x = A \sin z + C \cos y,$$

$$\frac{dy}{dt} = B_y = B \sin x + A \cos z, \qquad (4.12)$$

$$\frac{dz}{dt} = B_z = C \sin y + B \cos x,$$

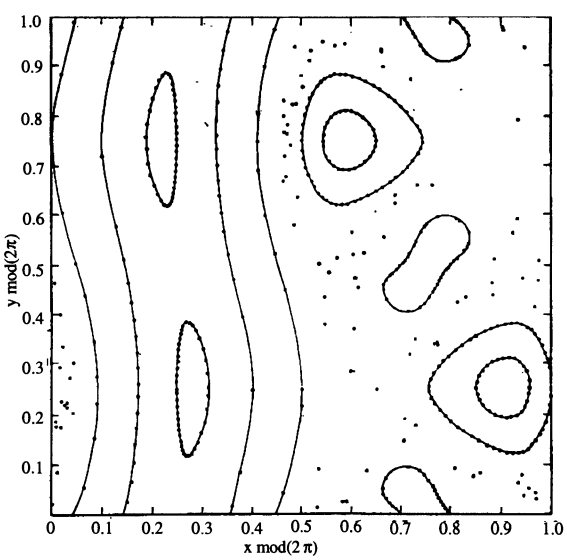

Figure 4.4. Numerical integration of Equations (4.12). The points joined by a curve correspond to the same field line, and the isolated points correspond to a single field line not constrained to a surface. Adapted from: M. Hénon, *C.R. Acad. Sci. Paris* **262A**, 312 (1966).

where A, B and C are constants. By taking the curl, the field $[B_x, B_y, B_z]$ is readily verified to be force-free with $\alpha = 1$. The system is periodic with period 2π. Hénon computed the flow given by Eqs. (4.12) on a 3-dimensional torus, T^3. The topology of T^3 is obtained by identifying the opposite faces of a cube; i.e., here by using the coordinates x (mod 2π), y (mod 2π), and z (mod 2π). Fig. 4.4 shows the numerical integration done by Hénon for $A = \sqrt{3}$, $B = \sqrt{2}$, and $C = 1$. The points correspond to

successive intersections of a field line with the x,y - plane (mod 2π) specified by the section $z = 0$ (mod 2π). The points joined by a curve correspond to the same field line, while all the isolated points also correspond to a single field line that is not constrained to a surface. The latter are "semi-ergodic," filling a region of 3-dimensional space. Dombre, et al.[11] have made an extensive study of this system of equations.

Fig. 4.5 shows the behavior of Eqs. (4.12), with Hénon's values for A, B and C, in three dimensions. The field lines in this figure are constrained to different surfaces depending on the choice of initial conditions. Dombre, et al. showed there are six such surfaces or principal vortices.

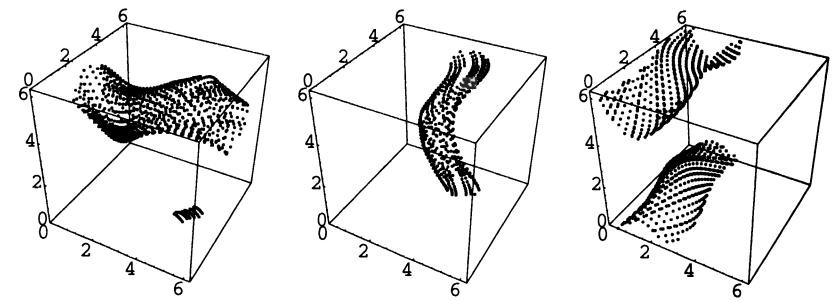

Figure 4.5. The behavior of Eqs. (4.12) depending on different initial conditions. The field lines are constrained to different surfaces. There are six such surfaces or principal vortices. The topology is that of a 3-dimensional torus T^3 where opposite faces of the cube are identified.

When either of the parameters A, B, or C vanishes, Eqs. (4.12) describing the flow are separable and integrable, and the solution can be written in terms of elliptic integrals.

Fig. 4.6 shows two views of a far more complicated behavior obtained by yet another set of initial conditions. Here, outside the principal vortices, the field lines show a chaotic behavior. The figures were produced by using the Runge-Kutta method of numerically integrating systems of ordinary, first-order differential equations.

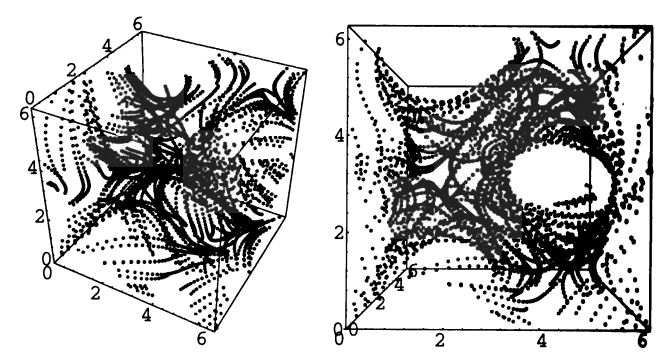

Figure 4.6. The behavior of Eqs. (4.12) for another set of initial conditions. Outside the principal vortices, the field lines have a chaotic character.

4.2 Magnetic Field Helicity

The helicity of a magnetic field can be expressed as

$$\mathcal{H} = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\ \mathrm{d}V. \tag{4.13}$$

It is a pseudoscalar that is gauge invariant in simply connected domains provided the normal component of $\boldsymbol{B}$ vanishes on the bounding surface of V. In multiply connected domains, such as the volume bounded by a toroidal magnetic surface, the meaning of the integral, and its physical interpretation, is not immediately clear because of the presence of the vector potential $\boldsymbol{A}$. For the latter case, consider a gauge transformation $\boldsymbol{A} \rightarrow \boldsymbol{A} + \nabla\chi$. The helicity introduced by the term associated with $\nabla\chi$ is

$$\mathcal{H} = \int_V \nabla\chi\cdot\boldsymbol{B}\ \mathrm{d}V = \int_V \nabla\cdot(\chi\boldsymbol{B})\ \mathrm{d}V = \int_{\partial V} (\chi\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{S} + \int_{\Sigma_1} [\chi]\boldsymbol{B}\cdot\mathrm{d}\Sigma_1, \tag{4.14}$$

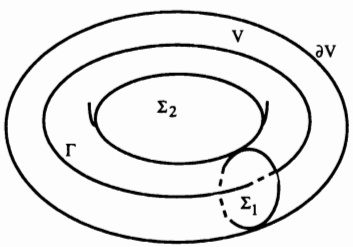

Figure 4.7. Σ_1 is the cut need to make the scalar function χ single valued. The jump $[\chi]$ in the value of χ across the surface Σ_1 is constant on Σ_1, so that Γ may be taken as bounding Σ_2.

where $[\chi]$ is the jump across Σ_1, the cut needed to make χ single valued (see Fig. 4.7). Since the bounding surface of the torus is a magnetic surface, the first term on the right hand side vanishes. It can be shown[12] [see Chapter 5] that $[\chi]$ is constant on Σ_1. Thus, Γ can be taken as bounding Σ_2 and $[\chi]$ can be taken outside the integral.

Now for any two points p_1 and p_2, the difference in the value of the scalar function χ is

$$\chi_{p2} - \chi_{p1} = \int_{p1}^{p2} \nabla\chi\cdot\mathrm{d}\boldsymbol{l}. \tag{4.15}$$

If the path of integration is closed, but does not cross the cut needed to make χ single valued, the integral will vanish by Stokes' theorem. When the path of integration Γ does cross the cut Σ_1, the value of the jump $[\chi]$ is

$$[\chi] = \oint_\Gamma \nabla\chi\cdot\mathrm{d}\boldsymbol{l} = \int_{\Sigma_2} \boldsymbol{B}\cdot\mathrm{d}\Sigma_2 = \Phi_{\Sigma_2}, \tag{4.16}$$

where Φ_{Σ_2} is the magnetic flux through the surface Σ_2. Thus,

$$\mathcal{H} = \int_{\Sigma_1} [\chi]\boldsymbol{B}\cdot\mathrm{d}\Sigma_1 = \oint_\Gamma \nabla\chi\cdot\mathrm{d}\boldsymbol{l} \int_{\Sigma_1} \boldsymbol{B}\cdot\mathrm{d}\Sigma_1 = \Phi_{\Sigma_1}\Phi_{\Sigma_2}. \tag{4.17}$$

The helicity introduced by the gauge transformation is therefore seen to explicitly depend on the topology. Physically, the helicity derived from the gauge term corresponds to the linkage of the flux contained within the torus with that passing through the "hole in the donut."

4.2.1. Twist, Kink and Link Helicity

Using a geometric invariant of a space curve, called the writhing number[13], Berger and Field[14] decomposed the helicity of a magnetic field into the sum of "twist" and "kink" helicities and, based on the work of Fuller,[15] defined the helicity of open field structures. They made use of a theorem[16] from knot theory[17] that states that the linking number of two curves X and Y without common points can be written as the sum of the twist number T_W and the writhing number W_R,

$$L_{XY} = T_W + W_R. \tag{4.18}$$

This makes sense, for example, if one considers X and Y to be the edges of a ribbon; the example given by Berger and Field identifies X with the central axis of a flux rope and Y with a field line winding about this axis. The topological interpretation of helicity in terms of the Gauss linking number and its limiting form, the Călugăreanu invariant, has been extensively discussed by Moffatt and Ricca.[18]

In knot theory, a crossover is defined to be positive or negative by the right hand rule (see Fig. 4.8). The writhing number, W_R, is defined to be the sum $(N_+ - N_-)$, where N_+ and N_- are the number of positive and negative crossovers. If a link is projected onto a plane so that the crossovers are at points p, the linking number is defined as

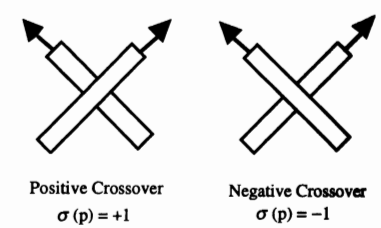

Figure 4.8. Positive and negative crossovers as determined by the right hand rule.

$$L_{XY} = \frac{1}{2} \sum_{p \in X \cap Y} \sigma(p), \tag{4.19}$$

where $X \cap Y$ denotes the set of crossings of the link X with the link Y, and $\sigma(p)$ corresponds to the sign, defined by the right hand rule, assigned to each crossover . Examples are given in Fig. 4.9.

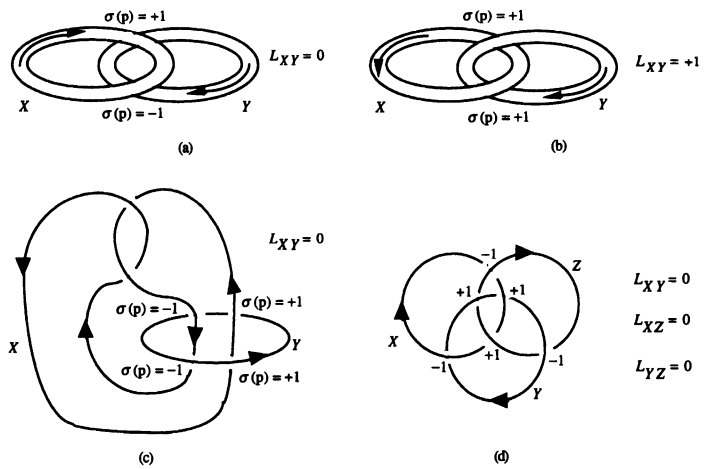

Figure 4.9. Examples of linking number as defined by Eq. (4.19). The absence of linkage implies that the linking number vanishes, but the converse is not true as seen in (c).

Note that the absence of linkage implies that the linking number vanishes, but the converse is not true as illustrated by the Whitehead link shown in Fig. 4.9(c). While the Borromean rings shown in Fig. 4.9(d) are clearly linked, any two components are unlinked. The twist number, T_W, will be limited here to integer values and the total twist will generally be multiples of 2π.

Assume that the writhing number is an invariant. As is seen in Fig. 4.10, the effect of switching a positive crossing to a negative crossing is equivalent to inserting a link on one of the elements of the crossing.

In what follows, uniformly twisted flux tubes will be used for heuristic purposes. It must be understood, however, that such tubes, where the $\hat{z}$ and $\hat{\phi}$ components of the field are constant *and no current is present*, cannot exist in nature. This is clear from the fact that an axial current *must* be present since $\oint \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l}$, where the integral is around the tube, does not vanish. Thus, twisted flux loops will have currents present that will in general give rise to additional magnetic fields that link the twisted loop.

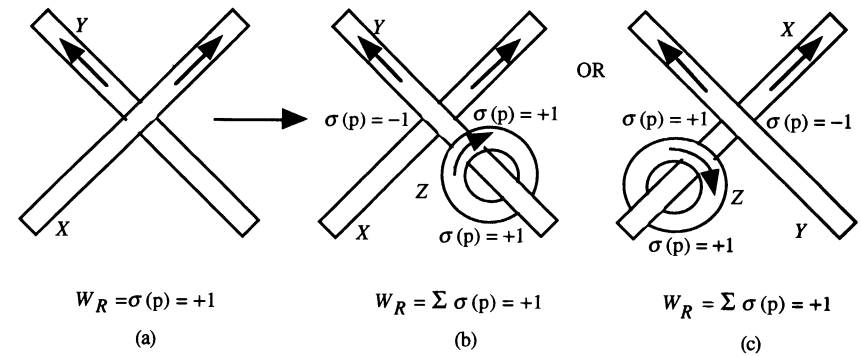

Figure 4.10. Switching a positive crossing to a negative crossing is equivalent to inserting a link on one of the elements of the crossing. In (a), $p \subset (X \cap Y)$; in (b), $p \subset (X \cap Y) \cup (Y \cap Z)$; and in (c), $p \subset (X \cap Y) \cup (X \cap Z)$.

For a flux tube of small cross section, the integrand in Eq. (4.13) can be written as $\boldsymbol{A} \cdot \boldsymbol{B}\, \mathrm{d}S\, \mathrm{d}l = \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} B\, \mathrm{d}S$, where $\mathrm{d}\boldsymbol{l}$ is along the axis of the flux tube. Thus, if an originally untwisted flux tube, assumed to be a torus of circular cross section carrying a flux Φ, is cut, given a twist of 2π, and reconnected, the helicity introduced by the twist is

$$\mathcal{H}_T = \Phi \oint \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{l} = \Phi^2. \tag{4.20}$$

Note that this is essentially the same result as obtained in Eq. (4.17). If the original twist had been 4π, the result would be $2\Phi^2$, and in general, a twist of $2\pi T_W$ corresponds to a twist helicity of $\mathcal{H}_T = T_W \Phi^2$.

The helicity associated with the writhing number is obtained by observing that a torus with twist number ± 1 may be distorted into a figure-8 configuration which appears untwisted. The writhing number will no longer vanish, however, but will have value ± 1 depending on the sign of the original twist. Thus, $\mathcal{H}_K = W_R \Phi^2$.

The helicity associated with the linking number is $\mathcal{H}_L = 2L_{XY}\Phi^2$. Since the vanishing of the linking number does

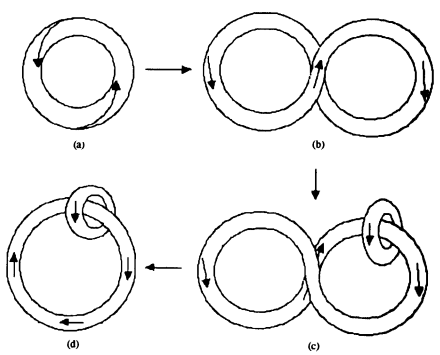

Figure 4.11. The sequence of manipulations that exchange a twist helicity of $2\Phi^2$ for a link helicity. (a) $H_T = 2\Phi^2$, $H_K = 0$, $H_L = 0$; (b) $H_T = \Phi^2$, $H_K = \Phi^2$, $H_L = 0$; (c) $H_T = \Phi^2$, $H_K = -\Phi^2$, $H_L = 2\Phi^2$; (d) $H_T = 0$, $H_K = 0$, $H_L = 2\Phi^2$.

not guarantee the absence of linkage [Fig. 4.9(c)], one might anticipate that there may be regions where the helicity vanishes even though the field lines are linked. An example of such a case is given in the next section.

The origin of the factor of two in the expression $\mathcal{H}_L = 2L_{XY}\Phi^2$ is interesting, and derives from the fact that a flux tube twisted by 4π ($\mathcal{H}_T = 2\Phi^2$) is equivalent to two linked but untwisted flux tubes. This can be seen as follows: Let the total conserved helicity be given by the sum of the twist helicity H_T, the kink helicity H_K, and the link helicity H_L. The twist helicity can be shared with a kink helicity as shown in Fig 4.11(b). In Fig. 4.11(c) the crossing is switched by introducing a link helicity; and in Fig. 4.11(d) the kink is eliminated by straightening out the figure eight, leaving only a link helicity. One can convince oneself of the validity of these exchanges by playing with a piece of ribbon.

To compute the helicity of the two linked flux tubes,[19] again replace the integrand of Eq. (4.13) with $\boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}S\mathrm{d}l = \boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}B\mathrm{d}S$, and note that the integral of $B\mathrm{d}S$ may be replaced with the flux Φ. Then $\Phi\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}$ integrated over one of the links gives Φ^2 since the flux linking the circuit is the flux Φ contained in the other link, and a similar integration over the second link gives another factor of Φ^2.

As a final example, consider a flux tube in the form of a trefoil knot, as shown in Fig. 4.12(a). Assume that initially only the kink helicity is different from zero. Figure 4.12(b) shows the exchange of kink helicity for link helicity, and Fig. 4.12(c) an additional exchange of kink helicity for twist helicity. Note that at each step the total helicity, consisting of the sum of twist, kink, and link helicities, is conserved.

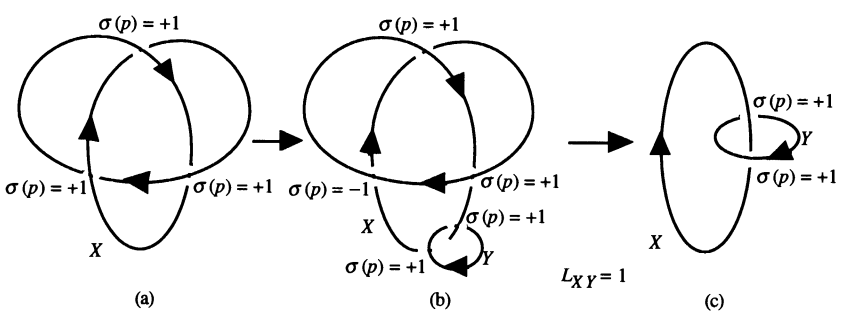

Figure 4.12. A trefoil knot: (a) $H_T = 0$, $H_K = 3\Phi^2$, $H_L = 0$; (b) shows an exchange of kink helicity for link helicity and $H_T = 0$, $H_K = \Phi^2$, $H_L = 2\Phi^2$; (c) shows a further exchange of kink helicity for twist helicity and $H_T = \Phi^2$, $H_K = 0$, $H_L = 2\Phi^2$.

4.2.2. *Helicity and Locality*

From the forgoing discussion, the physical interpretation of helicity as being a measure of the linkage of the field lines would seem to be on firm ground. But the

following example illustrates that this interpretation must not be taken too literally. Helicity can be non-localizable in the same sense that the magnetic energy of a magnetostatic field is not localizable. It will be shown that introducing a uniform twist in a previously untwisted flux tube does not increase the helicity in the flux tube itself, even though the field lines there are clearly linked. Instead, the twist creates a link helicity. Physically this results from the fact that the twist creates an axial current that is the source for a potential field that links the flux tube, thus giving rise to the link helicity.

A uniformly twisted flux tube, albeit with the $\hat{z}$ and $\hat{\phi}$ components of the field decreasing monotonically with increasing radius, can exist. As can be readily verified, the field[20,21]

$$\boldsymbol{B} = \left\{0, \frac{B_0}{(1+r^2/a^2)}\frac{r}{a}, \frac{B_0}{(1+r^2/a^2)}\right\}, \tag{4.21}$$

where a is a constant, satisfies the force-free field equations with α being given by,

$$\alpha = \frac{2}{a(1+r^2/a^2)}. \tag{4.22}$$

That the field is uniformly twisted can be seen from the fact that $B_\phi/B_z = r/a$. If the angle θ is defined with respect to the z-axis, $\tan\theta = B_\phi/B_z = r\,\mathrm{d}\phi/\mathrm{d}z$. Therefore the z-distance traversed is $z = a\int \mathrm{d}\phi$, independent of r. Note that the case of a constant axial field is given by $a \to \infty$.

In order to compute the helicity of the magnetic field given by Eq. (4.21), the vector potential must be determined. By symmetry, $\boldsymbol{A}$ is not a function of ϕ or z, and $A_r = 0$. $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ is readily integrated to give,

$$\begin{aligned} A_\phi &= \frac{B_0 a^2}{2r}\ln(a^2+r^2) \\ A_z &= -\frac{B_0 a}{2}\ln(a^2+r^2). \end{aligned} \tag{4.23}$$

The singular nature of these potentials is due to the fact that the current density (related to the field by the force-free relation) is not constrained to a finite volume. Such singular potentials often occur in magnetostatics.[22]

If one now uses Eq. (4.21) and Eqs. (4.23) to compute the integrand $\boldsymbol{A}\cdot\boldsymbol{B}$ of Eq. (4.13), it is found to vanish. In fact, the vanishing of the helicity does not require the explicit form of the potential given by Eqs. (4.23). The potential obtained by integrating $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ could be left in integral form and subsequently used to show that $A_\phi/A_z = -a/r$ (which means, incidentally, that the distance along the z-axis traversed by $\boldsymbol{A}$ in one turn about the axis is independent of r and is the same as that traversed by $\boldsymbol{B}$). Since $B_\phi/B_z = r/a$, $\boldsymbol{A}\cdot\boldsymbol{B} = (0, -A_z a/r, A_z)\cdot(0, B_z r/a, B_z) = 0$. Yet, the field lines are clearly linked and therefore the helicity cannot vanish.

The resolution to this apparent paradox lies in two additional factors: (i) The vector potential computed in Eqs. (4.23) is determined only up to the gradient of a scalar function $\nabla\chi$ which leads to an additional helicity term $\int \nabla\chi\cdot\boldsymbol{B}\,\mathrm{d}V$. (ii) Since α is not globally constant, the field lines lie on the surfaces α = constant, and these surfaces cannot be simply connected.

The solution given by Eq. (4.21) to the force-free field equation can be given the topology of a torus by identifying $z = \pm z_0$. The means of evaluating the additional term $\int \nabla\chi\cdot\boldsymbol{B}\,\mathrm{d}V$ when the volume of integration is a torus has been discussed in Section 4.2. The helicity introduced by this term is $\Phi_{\Sigma_1}\Phi_{\Sigma_2}$.

This example is interesting not only because the helicity explicitly depends on the topology, but without the gauge term, and the multiply-connected nature of the space, the helicity would vanish. It is also an explicit physical realization of the exchange of twist helicity for link helicity.

Akhmetiev and Ruzmaikin[23] have also discussed additional topological configurations that have zero helicity. An example of such a configuration given by these authors is the Whitehead link shown in Fig. 4.9(c). This is a non-trivial link with zero link helicity. Akhmetiev and Ruzmaikin construct a topological invariant for such a two flux tube configuration. The invariant is the self-linking number of the vector field $\boldsymbol{G} = \boldsymbol{A}_i\times\boldsymbol{A}_j - \phi_j\boldsymbol{B}_i + \phi_i\boldsymbol{B}_j$. Here $\boldsymbol{A}_i = \nabla\phi_i$ and $\boldsymbol{A}_j = \nabla\phi_j$ are the vector potentials (expressed as the gradients of scalar functions) outside the flux tubes U_i and U_j.

4.2.3. *The Classical and Asymptotic Hopf Invariant*

The interpretation of helicity as a measure of the linkage of the field lines can be demonstrated directly. But one should keep in mind the results of the last section: Requiring the boundary S of the volume of integration V to be a magnetic surface is not sufficient to determine the helicity associated with the field contained within the volume; the helicity may explicitly depend on the topology. The discussion begins by demonstrating the relation between helicity and the Gauss linking number.

In a simply connected domain, choose the Coulomb gauge where the vector potential satisfies $\nabla\cdot\boldsymbol{A} = 0$. Then each rectangular component of the vector potential will satisfy Poisson's equation, and the solution for $\boldsymbol{A}$ is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{\boldsymbol{J}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V'. \tag{4.24}$$

Since $\boldsymbol{J}(\boldsymbol{r}') = \nabla'\times\boldsymbol{H}(\boldsymbol{r}')$, the vector relation

$$\nabla'\times\frac{\boldsymbol{H}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|} = \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\nabla'\times\boldsymbol{H}(\boldsymbol{r}') + \nabla'\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\times\boldsymbol{H}(\boldsymbol{r}')$$

may be used to write the vector potential as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \nabla'\times\frac{\boldsymbol{H}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}V' + \frac{\mu_0}{4\pi}\int \boldsymbol{H}(\boldsymbol{r}')\times\nabla'\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\mathrm{d}V'. \tag{4.25}$$

Or, expressing the first integral as a surface integral,

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\int \hat{\boldsymbol{n}}\times\frac{\boldsymbol{B}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}\mathrm{d}S' + \frac{1}{4\pi}\int \boldsymbol{B}(\boldsymbol{r}')\times\nabla'\left(\frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\mathrm{d}V'. \tag{4.26}$$

For a bounded current distribution, as $\boldsymbol{r}'\to\infty$, $\boldsymbol{B}(\boldsymbol{r}')$ will fall off at least as fast as $1/r'^3$ so that the first integral vanishes far outside the distribution. Thus, performing the indicated gradient operation in the second integral, the vector potential may be written as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{4\pi}\int \boldsymbol{B}(\boldsymbol{r}')\times\frac{\boldsymbol{r}-\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'. \tag{4.27}$$

Using this result in the expression for the helicity given by Eq. (4.13) allows the helicity to be written as

$$\begin{aligned}\mathcal{H} &= \frac{1}{4\pi}\iint_{V,\,V'} \boldsymbol{B}(\boldsymbol{r})\cdot\frac{\boldsymbol{B}(\boldsymbol{r}')\times(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'\mathrm{d}V \\ &= \frac{1}{4\pi}\iint_{V,\,V'} \frac{[\boldsymbol{B}(\boldsymbol{r})\times\boldsymbol{B}(\boldsymbol{r}')]\cdot(\boldsymbol{r}-\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|^3}\mathrm{d}V'\mathrm{d}V.\end{aligned} \tag{4.28}$$

This form of the helicity integral makes it possible to explicitly show the connection between helicity and the linkage of the magnetic field lines.

Let the domains of integration be two untwisted flux filaments Γ and Γ' so that for each domain the field and volume element may be combined to give $\boldsymbol{B}(\boldsymbol{r})\mathrm{d}V = \boldsymbol{B}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{S}\mathrm{d}l = \delta\Phi\mathrm{d}\boldsymbol{l}$ and $\boldsymbol{B}(\boldsymbol{r}')\mathrm{d}V' = \boldsymbol{B}(\boldsymbol{r}')\cdot\mathrm{d}\boldsymbol{S}'\mathrm{d}l' = \delta\Phi'\mathrm{d}\boldsymbol{l}'$. Then,

$$\mathcal{H} = \frac{\Phi\Phi'}{2\pi}\iint_{\Gamma,\Gamma'} \frac{[d\boldsymbol{l}\times d\boldsymbol{l}']\cdot(\boldsymbol{l}-\boldsymbol{l}')}{|\boldsymbol{l}-\boldsymbol{l}'|^3} = 2n\Phi\Phi', \tag{4.29}$$

where $1/4\pi$ times the integral in Eq. (4.29) defines the Gauss linking number n. Equation (4.29) is the generalization of the link helicity discussed in Section 4.2.1.

The Gauss linking number is related to the degree of a mapping.[24] Let $\mathbb{M}^r$ and $\mathbb{N}^s$ be oriented closed manifolds in $\mathbb{E}^n$, with $r + s = n - 1$ and $\mathbb{M}^r \cap \mathbb{N}^s = 0$ so that the two manifolds have no points in common. Then $\mathbb{M}\times\mathbb{N}$ is an oriented manifold of dimension $n - 1$. Define the mapping $f:\mathbb{M}\times\mathbb{N} \to \mathbb{E}^n - \{0\}$ by $f(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y} - \boldsymbol{x}$, $\boldsymbol{x} \in \mathbb{M}$, $\boldsymbol{y} \in \mathbb{N}$ (since $\mathbb{M}$ and $\mathbb{N}$ have no points in common, $\boldsymbol{y} - \boldsymbol{x}$ never vanishes). Then the Gauss linking number is the degree of the mapping f, which counts the number of times $\mathbb{M}$ and $\mathbb{N}$ link. In Eq. (4.29) above, $n = 3$ and $r = s = 1$ so that the flux filaments are considered to be closed 1-dimensional manifolds. That the degree of f is always an integer is not obvious [see Spivak, Reference 1, p. 373].

The next section builds on these results by introducing the classical Hopf invariant which is closely related to the Gauss linking number. The following section discusses the asymptotic Hopf invariant, which is applicable to cases where the magnetic field may have a complicated topology and the field lines may not close. Equation (4.28) will prove to be important for the physical interpretation of the asymptotic Hopf invariant.

4.2.3.1 The Classical Hopf Invariant

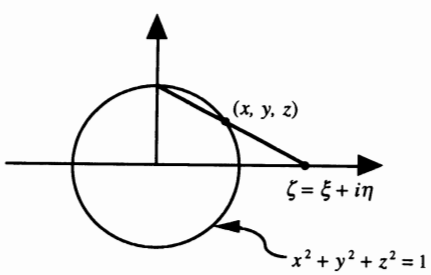

Figure 4.13. Stereographic projection of the complex plane (the one point compactification of $\mathbb{C} \cup \{\infty\}$).

There are several definitions of the Hopf invariant.[25] Perhaps the simplest, geometrically, is to consider a 3-sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ and define the Hopf map π of $\mathbb{S}^3$ onto the 2-sphere $\mathbb{S}^2$. If $\mathbb{S}^2$ is regarded as the one-point compactification $\mathbb{C} \cup \{\infty\}$ of the complex numbers, the specific identification of $\mathbb{S}^2$ and $\mathbb{C} \cup \{\infty\}$ can be achieved by use of a stereographic projection. Any point P on a line determined by the two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by $P = (1-\lambda)P_1 + \lambda P_2$, $\lambda \in \mathbb{R}$. With reference to Fig. 4.13, choose $P_1 = (0, 0, 1)$ and $P_2 = (\xi, \eta, 0)$. Then $x = \lambda\xi$; $y = \lambda\eta$; and $z = (1-\lambda)$. The constraint $x^2 + y^2 + z^2 = 1$ can be used to determine λ as

$$\lambda = \frac{2}{\xi^2 + \eta^2 + 1} = \frac{2}{1 + \zeta\bar{\zeta}}, \tag{4.30}$$

where $\zeta = \xi + i\eta$. x, y, and z are then given by

$$x = \frac{2\xi}{1+\zeta\bar{\zeta}}, \quad y = \frac{2\eta}{1+\zeta\bar{\zeta}}, \quad z = \frac{\zeta\bar{\zeta}-1}{\zeta\bar{\zeta}+1}$$
$$x + iy = \frac{2\zeta}{1+\zeta\bar{\zeta}}. \tag{4.31}$$

The sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ is given by the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. Identify a point on this sphere with two complex numbers (z_1, z_2), where $z_1 = x_1 + ix_2$ and $z_2 = x_3 + ix_4$. Then the equation of the sphere is $|z_1|^2 + |z_2|^2 = 1$; i.e., $\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}\times\mathbb{C} \mid |z_1|^2 + |z_2|^2 = 1\}$ The Hopf mapping $\pi : \mathbb{S}^3 \to \mathbb{S}^2$ is given by $\pi(z_1, z_2) = z_1/z_2$. To identify the image point on $\mathbb{S}^2$ of the Hopf mapping, set $\zeta = z_1/z_2$ in Eqs. (4.31). Then,

$$x + iy = 2z_1\bar{z}_2, \quad z = |z_1|^2 - |z_2|^2, \tag{4.32}$$

and,

$$\pi(z_1, z_2) \to \left[2\mathrm{Re}(z_1\bar{z}_2), 2\mathrm{Im}(z_1\bar{z}_2), |z_1|^2 - |z_2|^2\right] \in \mathbb{S}^2. \tag{4.33}$$

Now if $\lambda = e^{i\theta}$, $0 \le \theta \le 2\pi$, $\pi(\lambda z_1, \lambda z_2) = \pi(z_1, z_2)$. Therefore, $\pi^{-1}(u) \cong \mathbb{S}^1$ for any $u \in \mathbb{S}^2$. Given any two points $u, v \in \mathbb{S}^2$, the "circumferences" $\pi^{-1}(u)$ and $\pi^{-1}(v)$ are disjoint and linked in $\mathbb{S}^3$. This can be readily seen by choosing, for example, $u = (1, 0, 0)$ and $v = (-1, 0, 0)$ in $\mathbb{S}^2$ and using Eq. (4.33) to obtain the additional relations $2z_1\bar{z}_2 = 1$ for u and $2z_1\bar{z}_2 = -1$ for v. These are satisfied respectively by

$$z_1 = z_2 = \frac{e^{i\theta}}{\sqrt{2}} \quad \text{and} \quad z_1 = -z_2 = \frac{e^{i\theta}}{\sqrt{2}}, \tag{4.34}$$

where $z_1, z_2 \in \mathbb{S}^3$.

Visualizing the linkage is aided by using the stereographic projection $\Pi: \mathbb{S}^3 \to \mathbb{R}^3$, with the projection point, which will be chosen here to be $(0, 0, 0, -1) \in \mathbb{S}^3$, identified with the point at infinity. The procedure for doing the projection closely follows the discussion preceding Eq. (4.30). Fig. 4.13 applies with complex plane replaced by $\mathbb{R}^3$ and $\mathbb{S}^2$ replaced by $\mathbb{S}^3$. Since it is the inverse transformation that is of interest, choose $P_1 = (0, 0, 0, -1)$ and $P_2 = (x_1, x_2, x_3, x_4)$, both contained in $\mathbb{S}^3$. The relation $P = (1-\lambda)P_1 + \lambda P_2$, $\lambda \in \mathbb{R}$ then immediately determines λ to be $\lambda = 1/(1 + x_4)$. Thus, if the coordinates in $\mathbb{R}^3$ are given by $(\zeta_1, \zeta_2, \zeta_3)$, since $\zeta_i = \lambda x_i$,

$$\zeta_i = \frac{x_i}{1 + x_4}, \tag{4.35}$$

where $x_4 \neq -1$. Using this result and Eqs. (4.34) results in

$$\Pi[\pi^{-1}(u)]: \quad \zeta_1 = \zeta_3 = \frac{\cos\theta}{\sqrt{2}+\sin\theta}; \quad \zeta_2 = \frac{\sin\theta}{\sqrt{2}+\sin\theta}$$

$$\Pi[\pi^{-1}(v)]: \quad \zeta_1 = -\zeta_3 = \frac{\cos\theta}{\sqrt{2}-\sin\theta}; \quad \zeta_2 = \frac{\sin\theta}{\sqrt{2}-\sin\theta}. \tag{4.36}$$

If these curves are plotted as a function of θ in the mutually orthogonal planes $\zeta_1 = \pm\zeta_3$, one obtains two disjoint and singly linked ellipses.[26]

The Hopf mapping can be considered to be the projection of the Hopf bundle; i.e., of the locally trivial fiber space whose total space $\mathbb{S}^3$ has a base space $\mathbb{S}^2$ and fiber $\mathbb{S}^1$. A proof of this is given by Hu [Reference 23]. Note, however, that the sphere $\mathbb{S}^3$ is not homeomorphic to the direct product $\mathbb{S}^2\times\mathbb{S}^1$ since the fundamental groups of these spaces are not isomorphic.

For a general mapping $f: \mathbb{S}^{2n-1}\to\mathbb{S}^n$, $n \geq 2$, if u and v are distinct interior points of $\mathbb{S}^n$, then $f^{-1}(u)$ and $f^{-1}(v)$ are disjoint $(n-1)$-manifolds in $\mathbb{S}^{2n-1}$. The classical Hopf invariant is the linking number $\mathbb{L}[f^{-1}(u), f^{-1}(v)]$. A more formal definition can be given by using concepts from algebraic topology.[27] An n-chain is defined as a formal sum of n-simplices, a cycle is a chain whose boundary vanishes, and a fundamental cycle is one that is not a multiple of any other cycle. Now let u and v be contained in different n-simplices of $\mathbb{S}^n$ (see Fig. 4.14). If f is a simplicial map of $\mathbb{S}^{2n-1}$ onto $\mathbb{S}^n$ and C^n is an n-chain on $\mathbb{S}^{2n-1}$ bounded by the $(n-1)$-chain $f^{-1}(u)$, then $f(C^n)$ is an n-cycle on $\mathbb{S}^n$. Let S^n designate the fundamental n-cycle on $\mathbb{S}^n$. Then $f(C^n) = \gamma S^n$, where γ is the Hopf invariant. γ corresponds to the number of times the image $f(C^n)$ wraps around v. It is interesting to note that γ vanishes if n is odd.

That the Hopf invariant can be expressed as a volume integral was first shown by Whitehead.[28] This is most easily expressed in the language of differential forms. Let f be a mapping of $\mathbb{S}^{2n-1}\to\mathbb{S}^n$, $n \geq 2$, and let ω be an n-form on $\mathbb{S}^n$ such that $\int_{\mathbb{S}^n}\omega = 1$. Then there is a natural induced map f^* that maps n-forms on $\mathbb{S}^n$ to n-forms on $\mathbb{S}^{2n-1}$. The form $f^*\omega$ is a closed n-form on $\mathbb{S}^{2n-1}$ because $d(f^*\omega) = f^*d\omega = 0$, where $d\omega = 0$ since there are no $(n+1)$-forms on $\mathbb{S}^n$. Furthermore, since $\mathbb{S}^{2n-1}$ has no non-trivial n-dimensional cycles, there is an $(n-1)$-form, β, on $\mathbb{S}^{2n-1}$ such that $f^*\omega = d\beta$. The exterior product $\beta\wedge d\beta$ is then a $(2n-1)$-form on $\mathbb{S}^{2n-1}$. The Hopf invariant is

$$\gamma = \int_{\mathbb{S}^{2n-1}} \beta\wedge d\beta \tag{4.37}$$

It is readily shown that γ does not depend on either the choice of β, so long as $f^*\omega = d\beta$, or on the choice of ω.[29]

4.2.3.2 *The Asymptotic Hopf Invariant*

It was demonstrated in the last section that the classical Hopf invariant is equal to the linking number $\mathbb{L}[f^{-1}(u), f^{-1}(v)]$ of two disjoint $(n-1)$-manifolds in $\mathbb{S}^{2n-1}$ that are the preimages of two regular points $u, v \in \mathbb{S}^n$. The classical Hopf invariant is always an integer. The asymptotic Hopf invariant introduced by Arnold [Reference 7] can have any real value and can be identified in three dimensions with the helicity. The asymptotic Hopf invariant is defined as the mean value of the asymptotic linking number of a pair of field lines. If the field lines correspond to a unitary vector field that is tangent to the Hopf bundle, the asymptotic Hopf invariant and the classical Hopf invariant coincide.

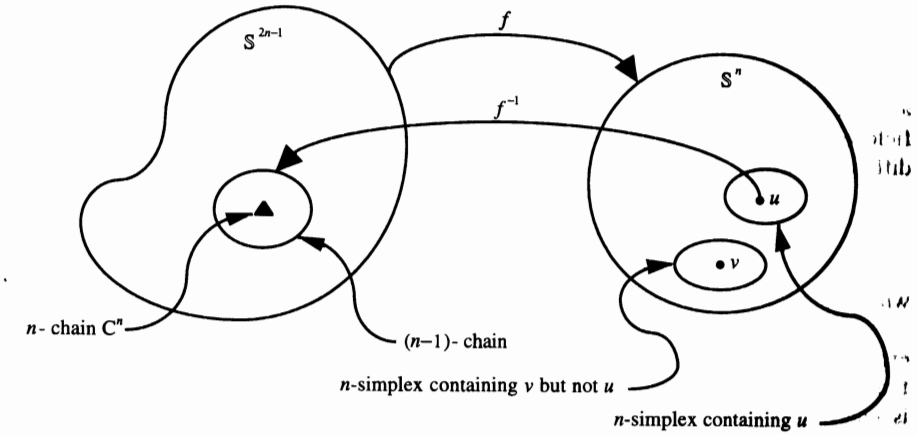

Figure 4.14. A mapping f of the $(2n-1)$-sphere $\mathbb{S}^{2n-1}$ onto the n-sphere $\mathbb{S}^n$. Here, $\partial C^n = f^{-1}(u)$. The Hopf invariant is the algebraic number of times $f(C^n)$ covers v and is the same as the linking number $\mathbb{L}[f^{-1}(u), f^{-1}(v)]$.

In Section 4.1.1 it was seen that if the field lines of a magnetic field constrained to the surface of a torus represented the trajectories of a uniform flow, it was possible for a given field line to be everywhere dense on the torus. Such a field line does not close, and therefore intersects the surface across the "hole in the donut" an arbitrary number of times. How does one compute the flux through this surface? The approach given below resolves such questions. For reasons of clarity, the discussion leaves out a number of mathematical subtleties that can be found in the literature cited.

Arnold uses the language of dynamical systems to introduce the concept of the asymptotic linking number. He then shows that the mean value of the asymptotic linking number corresponds to the asymptotic Hopf invariant. As will be seen, this is equivalent to giving an ergodic interpretation to the total helicity.

A dynamical system consists of a space $\mathbb{M}$ and a one-parameter mapping $g^t x$ of $\mathbb{M}$ into itself. If it is assumed that $g^0 x$ is the identity, and that $g^t[g^s x] = g^s[g^t x] = g^{t+s}x$, then g^t

defines a semiflow on $\mathbb{M}$. If the inverse $g^{-t}x$ is uniquely defined for all $x \in \mathbb{M}$, then g^t defines a flow, where both past and future behavior is unambiguously determined by the current state of the system. Henceforth, $\mathbb{M}$ will be assumed to be a simply connected 3-dimensional manifold. The phase velocity $v(x)$ at $x \in \mathbb{M}$ is defined as

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} g^t x = v(x). \tag{4.38}$$

Note that

$$v(g^\tau x) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=\tau} g^t x. \tag{4.39}$$

The similarity of Eq. (4.38) with Eqs. (4.3) suggests that $v(x)$ can be identified with the magnetic field. However, whether or not a given vector field uniquely determines a phase flow on a space $\mathbb{M}$ depends on whether or not the space is compact: Every vector field on a compact manifold is the phase velocity field of a one-parameter group of diffeomorphisms. Thus for a magnetic field $\boldsymbol{B}$ on a compact space,

$$\boldsymbol{B}(g^t\boldsymbol{x}) = \frac{\mathrm{d}}{\mathrm{d}t} g^t\boldsymbol{x}, \tag{4.40}$$

where the vector character of the quantities has been made explicit.

In the derivation of the asymptotic Hopf invariant, use is also made of Birkhoff's ergodic theorem from statistical mechanics. This theorem relates the time average of a function to its phase or state space average. Metrical indecomposability of the phase space is a necessary and sufficient condition for the equality (almost everywhere) of the time and phase averages.[30] The meaning of "almost everywhere" is that exceptions exist, but only on sets of measure zero.

The measure used for integration is defined on subsets of the phase space and must be invariant with respect to g^t. A measure μ is said to be ergodic if either $\mu(X) = 0$ or $\mu(X) = 1$, where X is an invariant subspace. A space having such a measure is said to be indecomposable. What this means is that since an invariant set corresponds to a set of complete trajectories, if an invariant set is separated into two sets, then either one of the component parts has measure zero, or both components are non-measurable.[31]

If $x \in X$ and X is metrically indecomposable, then Birkhoff's ergodic theorem can be written as

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(g^t x)\,\mathrm{d}t = \int_X f(x)\,\mathrm{d}\mu(x). \tag{4.41}$$

Birkhoff's theorem will be used to define the mean linking number with a curve, which will be seen to be equivalent to the flux through the surface bounded by the curve, and to

relate the asymptotic linking number of a pair of trajectories to the asymptotic Hopf invariant. In what follows, ξ is a divergence free vector field whose field lines are the phase flow $g^t: \mathbb{M}\to\mathbb{M}$, i.e., ξ replaces $\dot{x}$ in Eq. (4.38).

First, following Arnold, two equivalent definitions of the asymptotic linking number of the orbit $g^t x$, $x \in \mathbb{M}$, with a closed curve γ bounding a surface σ will be given. To do this, it is necessary to introduce a family of "short paths" $\Delta(x, y)$ that can be used to join any pair of points x, y of $\mathbb{M}$. It is assumed that if x, y do not belong to γ, then Δ does not intersect γ. As shown in Fig. 4.15, with $g^0 x = x$ and $g^T x = y$, a particular $\Delta(x, y)$ can be used to form a closed curve $\Gamma_T(x)$ from a segment of the orbit $g^t x$ corresponding to $0 \le t \le T$, where T is assumed to be a large number so that the orbit segment is large compared to Δ. Now let $N_T(x)$ denote the linking number of $\Gamma_T(x)$ with σ. Then the

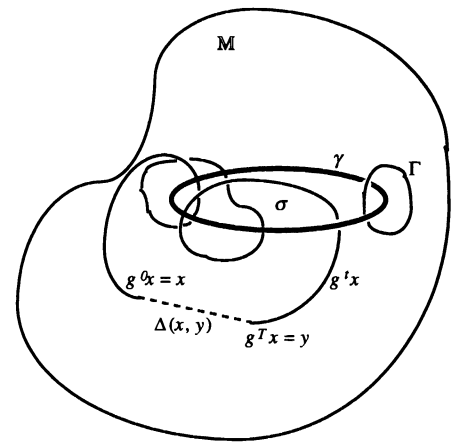

Figure 4.15. To define the asymptotic linking number of a single trajectory with a closed curve γ, short paths $\Delta(x, y)$ are introduced to form a closed curve from the segment of the orbit $g^t x$, $0 \le t \le T$.

asymptotic linking number, $\lambda(x)$, with the curve γ is given by

$$\lambda(x) = \lim_{T\to\infty} \frac{1}{T} N_T(x). \tag{4.42}$$

The second definition of the asymptotic linking number is obtained by constructing a closed 1-form α such that the linking number of every closed curve in $\mathbb{M}-\gamma$ with γ is given by the integral of α over that curve. Thus, $N_T(x)$ is replaced by an integral, and for the field ξ,

$$\hat{\lambda}(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \alpha\left(\frac{\mathrm{d}}{\mathrm{d}t} g^t x\right)\mathrm{d}t. \tag{4.43}$$

Noting that ξ is divergence free, it can be seen that the set of points asymptotic to γ under g^t form a set of measure zero. By proving that

$$\lim_{T\to\infty} \frac{1}{T}\int_{\Delta(g^T x,\, x)} \alpha(\xi)\mathrm{d}t = 0, \tag{4.44}$$

it is possible to show that $\lambda(x)$ is independent of the short paths Δ, and in the limit, $\hat{\lambda}(x) = \lambda(x)$ for almost all x.

The mean linking number of the phase flow $g^t x$ with γ is the average of the asymptotic linking number $\lambda(x)$ over $\mathbb{M}$,

$$\lambda = \int_{\mathbb{M}} \lambda(x)\nu, \tag{4.45}$$

where, following Arnold's notation, ν is the volume element of $\mathbb{M}$. Because $\lambda(x)$ is the time average of $\alpha(\xi)$, by Birkhoff's theorem the space average of $\alpha(\xi)$ must also be equal to λ. Thus,

$$\lambda = \int_{\mathbb{M}} \alpha(\xi)\nu. \tag{4.46}$$

Since $\alpha(\xi)$ is a real valued function evaluating the 1-form α over the vector field ξ, there is a unique vector a such that $\alpha(\xi) = (\xi, a) = (a, \xi)$. The inner product can also be written in terms of differential forms (which are discussed in somewhat greater detail in the next chapter): If β is the 1-form corresponding to the vector ξ, the Hodge star or duality operator allows the inner product between forms[32] to be defined by the relation $(\alpha\bullet\beta)\nu = \alpha\wedge *\beta$, where $*\beta$ is a 2-form corresponding to ξ. Thus, in terms of forms, λ may be written as

$$\lambda = \int_{\mathbb{M}} \alpha\wedge *\beta. \tag{4.47}$$

In truth, this integral is not over $\mathbb{M}$ but over $\mathbb{M}-\gamma$ which means the space is no longer simply connected. This feature is important for obtaining the interesting result that λ is equal to the flux of ξ through σ. To see this, remember that α is a closed form so that $\mathrm{d}\alpha = 0$, and that therefore there exists locally a function χ such that $\alpha = \mathrm{d}\chi$. χ can be made

single valued by taking the surface σ to be a cut. Since $\mathrm{d}(*\beta)$ is a 3-form, it can be written as the product of the divergence of ξ and the volume element. But since the divergence of ξ vanishes,

$$\lambda = \int_{\mathbb{M}-\gamma} \mathrm{d}\chi\wedge *\beta = \int_{\mathbb{M}-\gamma} \mathrm{d}(\chi *\beta) = \int_{\partial(\mathbb{M}-\gamma)} \chi *\beta + \int_{\sigma} [\chi] *\beta, \tag{4.48}$$

where $[\chi]$ is the jump in the function χ in crossing σ. Because ξ is divergence free, the integral of $*\beta$ over $\partial(\mathbb{M}-\gamma)$ vanishes. Since, as mentioned earlier, $[\chi]$ may be removed from under the integral, the last integral may be written out explicitly as,

$$\lambda = \int_{\Gamma} \mathrm{d}\chi \int_{\sigma} *\beta. \tag{4.49}$$

The second integral is the flux of ξ through σ, and the first corresponds to the number of times Γ in Fig. 4.15 is traversed, so that λ is a multiple of the flux of ξ through σ.

The discussion thus far has been concerned with the linking of the phase flow $g^t x$ with a closed curve γ. Consider now a pair of points $x_1, x_2 \in \mathbb{M}$ and the phase flows $g^{t_i}x_i$, $i = 1,2$. The approach taken above can be generalized to address the asymptotic linking of the trajectories $g^{t_i}x_i$. If ξ is identified with $\boldsymbol{B}$, this will result in Eq. (4.28), thereby showing that the average value of the asymptotic linking number of a pair of trajectories is equal to the asymptotic Hopf invariant defined in the next paragraph.

The asymptotic Hopf invariant is defined for a vector field ξ that is "homologous to zero." The meaning of this phrase is the subject of De Rham's theorems discussed in Chapter 5, but can be summarized as follows: if β is the 1-form corresponding to a divergence free vector ξ, $*\beta$ is a 2-form; $\mathrm{d}(*\beta)$ is a 3-form which vanishes since ξ is divergence free; since $\mathrm{d}(*\beta) = 0$, a 1-form potential α exists locally such that $*\beta = \mathrm{d}\alpha$, i.e., $*\beta$ is locally exact; for α to exist globally, the closed surface periods of $*\beta$ must vanish over any 2-cycle (here, essentially a closed surface). If $*\beta$ corresponds to a magnetic field, this is guaranteed by the non-existence of magnetic monopoles. Thus, if ξ is a divergence free vector field homologous to zero, the asymptotic Hopf invariant is defined in terms of the 1-form α as

$$\mathrm{I}(\xi) = \int_{\mathbb{M}} \alpha\wedge\mathrm{d}\alpha. \tag{4.50}$$

When $\mathrm{d}\alpha = *\beta$ corresponds to a magnetic field, α represents the vector potential and this integral is the helicity given in Eq. (4.13).

Returning to the asymptotic linking of the trajectories $g^{t_i}x_i$ ($i = 1,2$), short paths $\Delta(g^{T_i}x_i, x_i)$, are again introduced to close the orbit segments $g^{t_i}x_i$ for $0 \le t_i \le T_i$, where the T_i are chosen to be large numbers. The resulting closed curves $\Gamma_{T_i}(x_i)$ are shown in Fig. 4.16.

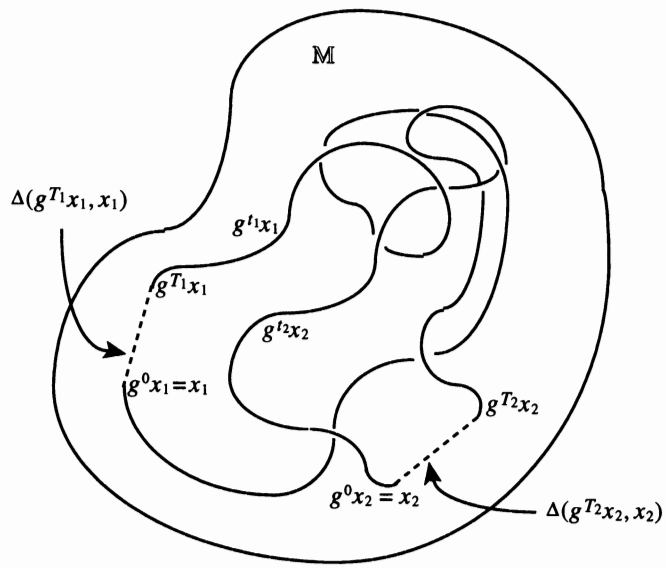

Figure 4.16. To define the asymptotic linking number of a pair of trajectories, closed curves are formed by introducing a system of short paths $\Delta(g^{T_i}x_i, x_i)$.

If $N_{T_1,T_2}(x_1, x_2)$ is the linking number of the closed curves $\Gamma_{T_i}(x_i)$, the asymptotic linking number of the pair of closed trajectories is defined as the limit

$$\lambda(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{N_{T_1,T_2}(x_1, x_2)}{T_1 T_2}, \tag{4.51}$$

where it is assumed that the parameters T_i vary so that the closed trajectories $\Gamma_{T_i}(x_i)$ do not intersect.

The form of the Gauss linking number identified in Eq. (4.29) suggests that a second form of the asymptotic linking number of the pair of closed trajectories be defined as

$$\hat{\lambda}(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{T_1 T_2} \frac{1}{4\pi} \int\int_0^{T_2, T_1} \frac{\xi_1 \cdot (\xi_2 \times \{x_1(t_1) - x_2(t_2)\})}{|x_1(t_1) - x_2(t_2)|^3} \, dt_1 dt_2 , \tag{4.52}$$

where

$$\xi_i = \frac{d}{dt_i}\bigg|_{t_i=0} g^{t_i} x_i , \quad i = 1,2. \tag{4.53}$$

It can again be shown that, for almost all x_i, the difference $\lambda(x_1, x_2) - \hat{\lambda}(x_1, x_2)$ vanishes in the limit that $T_1, T_2 \to \infty$, and that the limit is independent of the system of short paths.

The average value of the asymptotic linking number of a pair of trajectories is then defined as

$$\lambda = \iint_{M \times M} \lambda(x_1, x_2) v_1 v_2 . \tag{4.54}$$

Using Birkhoff's ergodic theorem, the time average can be replaced by a space average,[33] so that λ can be written as

$$\lambda = \frac{1}{4\pi} \iint_M \frac{\xi_1 \cdot (\xi_2 \times \{x_1(t_1) - x_2(t_2)\})}{|x_1(t_1) - x_2(t_2)|^3} v_1 v_2 . \tag{4.55}$$

If ξ is identified with $\boldsymbol{B}$, this is the same as Eq. (4.28) for the helicity. *Thus, the helicity can be identified with the average value of the asymptotic linking number of a pair of trajectories and is equivalent to the asymptotic Hopf invariant.*

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Chapter 5

MAGNETIC ENERGY IN MULTIPLY CONNECTED DOMAINS

The total magnetic field energy associated with currents $\boldsymbol{J}$ in a volume V is given by,

$$E = \frac{1}{2}\int_V \boldsymbol{J}\cdot\boldsymbol{A}\,\mathrm{d}V. \tag{5.1}$$

By taking the dot product of the force-free field equation $\nabla\times\boldsymbol{H} = \alpha(x)\boldsymbol{H}$ with $\boldsymbol{A}$ and integrating,[1] Eq. (5.1) implies that

$$E = \frac{1}{2}\int_V \boldsymbol{H}\cdot(\nabla\times\boldsymbol{A})\mathrm{d}V + \frac{1}{2}\int_S (\boldsymbol{H}\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{S} = \frac{1}{2}\int_V \alpha\boldsymbol{A}\cdot\boldsymbol{H}\,\mathrm{d}V, \tag{5.2}$$

where S is the bounding surface of the volume V. The first equality in Eq. (5.2) is true in general, while the second is true for force-free fields. For constant α fields, Yang[2] has noted that the magnetic energy can be written as the product of α and the total magnetic helicity.

If α is not assumed to be constant, the decomposition of the magnetic energy into the product of α and the total magnetic helicity is not possible. There is, however, a general relationship between the energy and helicity that is derivable from a combination of the Cauchy-Schwarz inequality ($|\boldsymbol{A}\cdot\boldsymbol{B}|\le|\boldsymbol{A}||\boldsymbol{B}|$) and a special case of the Hölder inequality for integrals (also known as the Schwarz inequality). The latter combination allows the following inequality to be written for the helicity in a domain D,

$$\int_D A^2\,\mathrm{d}V \int_D B^2\,\mathrm{d}V \geq \left[\int_D \boldsymbol{A}\cdot\boldsymbol{B}\;\mathrm{d}V\right]^2 = \mathcal{H}^2. \tag{5.3}$$

The equality between the integrals holds if and only if $|\boldsymbol{B}| = \alpha|\boldsymbol{A}|$, where α is a positive constant. This will be true for force-free fields with constant α (assuming α is positive) since, in this case, $\nabla\times\boldsymbol{B} = \alpha\boldsymbol{B} = \alpha\nabla\times\boldsymbol{A} = \nabla\times\alpha\boldsymbol{A}$. Thus, $\nabla\times(\boldsymbol{B}-\alpha\boldsymbol{A}) = 0$, which implies that $\boldsymbol{B} = \alpha\boldsymbol{A}+\nabla\chi$. So, $|\boldsymbol{B}| = \alpha|\boldsymbol{A}|$ will hold if it is assumed that the arbitrary gradient vanishes. A lower bound for the left hand side of Eq. (5.3) can be established by considering the variation,

$$\int_D \delta\boldsymbol{A}\cdot\boldsymbol{A}\;\mathrm{d}V \int_D B^2\,\mathrm{d}V + \int_D A^2\,\mathrm{d}V \int_D (\nabla\times\boldsymbol{A})\cdot(\nabla\times\delta\boldsymbol{A})\,\mathrm{d}V = 0. \tag{5.4}$$

This means that $\int_D \delta\boldsymbol{A}\cdot\boldsymbol{A}\;\mathrm{d}V = 0$ for all variations such that $\int_D (\nabla\times\boldsymbol{A})\cdot(\nabla\times\delta\boldsymbol{A})\,\mathrm{d}V = 0$.

Using the vector relation $\nabla\cdot[(\nabla\times\boldsymbol{A})\times\delta\boldsymbol{A}) = \delta\boldsymbol{A}\cdot[\nabla\times(\nabla\times\boldsymbol{A})] - (\nabla\times\boldsymbol{A})\cdot(\nabla\times\delta\boldsymbol{A})$, the last of the integrals in Eq. (5.4) may be written as

$$\begin{aligned}\int_D (\nabla\times\boldsymbol{A})\cdot(\nabla\times\delta\boldsymbol{A})\,\mathrm{d}V &= \int_D \delta\boldsymbol{A}\cdot[\nabla\times(\nabla\times\boldsymbol{A})]\,\mathrm{d}V \\ &\quad - \int_D \nabla\cdot[(\nabla\times\boldsymbol{A})\times\delta\boldsymbol{A}]\,\mathrm{d}V.\end{aligned} \tag{5.5}$$

But, by Stokes' theorem, the last integral in Eq. (5.5) is $\int_{\partial D} [(\nabla\times\boldsymbol{A})\times\delta\boldsymbol{A}]\cdot\mathrm{d}\boldsymbol{S}$. If one assumes that the system is closed, this integral will vanish since $\delta\boldsymbol{A} = 0$ on ∂D for a closed system.[3] Thus, $\int_D \delta\boldsymbol{A}\cdot\boldsymbol{A}\;\mathrm{d}V = 0$ for all variations such that $\int_D \delta\boldsymbol{A}\cdot[\nabla\times(\nabla\times\boldsymbol{A})]\,\mathrm{d}V = 0$. Consider $\int_D \delta\boldsymbol{A}\cdot\boldsymbol{A}\;\mathrm{d}V = 0$ to be a constraint. One may then introduce a constant Lagrange multiplier q^2, to write

$$\int_D \delta\boldsymbol{A}\cdot[\nabla\times(\nabla\times\boldsymbol{A}) - q^2\boldsymbol{A}]\,\mathrm{d}V = 0. \tag{5.6}$$

Since $\delta\boldsymbol{A}$ is arbitrary, Eq. (5.6) implies that $\nabla\times\boldsymbol{B} - q^2\boldsymbol{A} = 0$. If the latter equation is dotted with $\boldsymbol{A}$ and integrated over the domain D,

$$\int_D (\nabla\times\boldsymbol{B})\cdot\boldsymbol{A}\;\mathrm{d}V - q^2\int_D A^2\,\mathrm{d}V = 0. \tag{5.7}$$

Note that the left hand side of this equation is $2\mu_0$ times the energy. Multiplying each side of Eq. (5.7) by $\int_D B^2\,\mathrm{d}V$, and using the inequality of Eq. (5.3), one finds that

$$\mathcal{H}^2 \leq \frac{2\mu_0 E}{q^2}\int_D B^2\,\mathrm{d}V = \frac{1}{q^2}\int_D \alpha\boldsymbol{A}\cdot\boldsymbol{B}\;\mathrm{d}V\int_D B^2\,\mathrm{d}V, \tag{5.8}$$

where the inequality is general, and the equality on the right hand side only holds for force-free fields.

Now $\nabla\times(\nabla\times\boldsymbol{A}) - q^2\boldsymbol{A} = 0$ can be written as $\nabla(\nabla\cdot\boldsymbol{A}) - \nabla^2\boldsymbol{A} - q^2\boldsymbol{A} = 0$. With the constraint that $\nabla\cdot\boldsymbol{A} = 0$, one obtains the eigenvalue equation $\nabla^2\boldsymbol{A} + q^2\boldsymbol{A} = 0$. Let q_0 be the smallest eigenvalue. Then Eq. (5.7) implies that

$$2\mu_0 E \geq q_0^2\int_V A^2\mathrm{d}V. \tag{5.9}$$

If it is assumed that the boundary term of Eq. (5.2) vanishes, Eq. (5.9) can be written as

$$\int_V B^2\mathrm{d}V \geq q_0^2\int_V A^2\mathrm{d}V. \tag{5.10}$$

When combined with the inequality of Eq. (5.3), this gives the result[4]

$$\int_V B^2\mathrm{d}V \geq q_0\mathcal{H}. \tag{5.11}$$

Consider the right hand side of Eq. (5.8) which applies to general force-free fields. Again let q_0 be the smallest eigenvalue. $\nabla\times\boldsymbol{B} - q_0^2\boldsymbol{A} = 0$ implies that $\alpha\boldsymbol{B} - q_0^2\boldsymbol{A} = 0$. If the force-free field has minimum energy, α is a constant so that by taking the curl of the latter equation, $\alpha^2\boldsymbol{B} - q_0^2\boldsymbol{B} = 0$. Thus, $\alpha^2 = q_0^2$. So for minimum energy fields,

$$\int_V B^2\mathrm{d}V \geq \alpha\mathcal{H}. \tag{5.12}$$

It is clear that the equality only holds in the case where the boundary term of Eq. (5.2) vanishes.

From Eq. (5.2), it can be seen that both the volume and surface integrals contribute to the magnetic energy. The usual textbook discussion of the surface integral in Eq. (5.2) takes the bounding surface of the configuration to be at infinity. Because the source currents are assumed to occupy a finite, closed volume, and since $\boldsymbol{H}$ falls off at least as fast as r^{-2} and $\boldsymbol{A}$ as r^{-1}, it is argued that the integral vanishes as r^{-1} at a minimum. However, if S is a magnetic surface of finite dimension, this argument does not hold and the integral cannot be ignored. That this integral must play a role in the energy of finite force-free fields has been noted by Reiman,[5] who pointed out that if account is not taken of this boundary term, one obtains an incorrect expression for the energy of a force-free state. If the currents are confined to a volume V_c, and if the whole space, V_∞, is simply connected, the physical interpretation of the boundary term is readily understood by observing that it is identically equal to $\int_{V_\infty - V_C} \boldsymbol{B}\cdot\boldsymbol{H}\, \mathrm{d}V$ transformed by Stokes' theorem to a surface integral:

$$\int_{V_\infty - V_C} \boldsymbol{B}\cdot\boldsymbol{H}\, \mathrm{d}V = \int_{V_\infty - V_C} \boldsymbol{H}\cdot(\nabla\times\boldsymbol{A})\, \mathrm{d}V$$

$$= \int_{V_\infty - V_C} [\nabla\cdot(\boldsymbol{A}\times\boldsymbol{H}) + \boldsymbol{A}\cdot(\nabla\times\boldsymbol{H})]\mathrm{d}V \tag{5.13}$$

$$= \int_{\partial(V_\infty - V_C)} (\boldsymbol{A}\times\boldsymbol{H})\cdot\mathrm{d}\boldsymbol{S} + \int_{V_\infty - V_C} \boldsymbol{A}\cdot\boldsymbol{J}\, \mathrm{d}V,$$

where ∂V means the boundary of V. The last integral vanishes since the currents vanish outside V_c. Because the fields vanish at infinity, the boundary of interest is $\partial(V_\infty - V_C) = -\partial V_C$, so that

$$\int_{\partial(V_\infty - V_C)} (\boldsymbol{A}\times\boldsymbol{H})\cdot\mathrm{d}\boldsymbol{S} = -\int_{\partial V_C} (\boldsymbol{A}\times\boldsymbol{H})\cdot\mathrm{d}\boldsymbol{S} = \int_{\partial V_C} (\boldsymbol{H}\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{S}. \tag{5.14}$$

Thus, for simply connected spaces, the boundary term is the energy in the field exterior to V_c due to currents within V_c. But, as discussed in Chapter 4, if the magnetic field is not singular, finite magnetic surfaces -- such as those found to be associated with the solutions to the force-free field equations discussed in Chapter 3 -- cannot be simply connected.

To discuss the magnetic field energy in multiply connected domains, it will be useful to recast Eqs. (5.1) and (5.2) into the language of differential forms in 3-dimensional Euclidean space. This will facilitate contact with De Rham's theorems which will be used to give a general interpretation of the boundary term. To that end, replace the vector potential $\boldsymbol{A}$ with a 1-form A. Then the magnetic field $\boldsymbol{B}$ is given by the 2-form $B = \mathrm{d}A$ (equivalent to the local statement $\boldsymbol{B} = \nabla\times\boldsymbol{A}$). Taking the exterior derivative of B shows that B is a closed 2-form; i.e., $\mathrm{d}B = 0$ (corresponding to $\nabla\cdot\boldsymbol{B} = 0$). B is also *locally* an exact form since $B = \mathrm{d}A$. The question of the existence of A *globally* is the subject of De Rham's theorems.

Expressing Eqs. (5.1) and (5.2) in the language of differential forms also requires the use of the Hodge $*$ or duality operator introduced in Chapter 4. Here, since the application is to 3-dimensional space, the duality operator maps p-forms onto $(3-p)$-forms. Note that this operator has the property that the subspace of $*\omega$ is orthogonal to that of ω, where ω is a p-form. Although the $*$ operator is defined locally, it is independent of local coordinates, but does depend on the existence of an inner product and the orientation of the space.

The helicity is now readily expressed in terms of differential forms as

$$\mathcal{H} = \int_V A\wedge B. \tag{5.15}$$

If the flux across every closed surface in the domain V vanishes, this is the asymptotic Hopf invariant discussed in Chapter 4. If the current $\boldsymbol{J}$ is defined as the 1-form J, the 2-form $*J$ can be used to write the differential form of Ampere's circuital law, $\mu_0 *J = \mathrm{d}(*B)$. The magnetic energy, Eq. (5.1), can be written as,

$$\mu_0 E = \frac{1}{2}\int_V A\wedge\mathrm{d}(*B). \tag{5.16}$$

Using Stokes' theorem and the force-free relation $\mathrm{d}(*B) = \alpha B$, Eq. (5.2) becomes

$$\mu_0 E = \frac{1}{2}\int_V B\wedge *B - \frac{1}{2}\int_{\partial V} A\wedge *B = \frac{1}{2}\int_V \alpha A\wedge B. \tag{5.17}$$

Notice that if the inner product on p-forms is defined as

$$\langle\alpha,\beta\rangle = \int_V \alpha\wedge *\beta, \tag{5.18}$$

and one introduces the co-differential operator given in 3-dimensional space by $\delta\omega = (-1)^p *\mathrm{d}(*\omega)$, ω being a p-form, the boundary term can be written as[6]

$$\int_{\partial V} A\wedge *\mathrm{d}A = \langle B,B\rangle - \langle A,\delta \mathrm{d}A\rangle. \tag{5.19}$$

This shows that unless the boundary term vanishes, d and δ will not be adjoint operators as is usually the case for closed manifolds. Note also that the co-differential operator can be used to write the differential form of Ampere's circuital law as $\mu_0 J = \delta B$ since $** = +1$ applied to 1-forms.

The co-differential operator δ allows the Laplacian operator to be written as $\Delta = \mathrm{d}\delta + \delta\mathrm{d}$; it maps p-forms onto p-forms. This form of the Laplacian is of interest because of the Hodge decomposition theorem, which states that if ω is a p-form on a compact oriented manifold, there is a unique $(p-1)$-form α, a $(p+1)$-form β and a harmonic p-form γ (meaning that it satisfies $\Delta\gamma = 0$) such that $\omega = \mathrm{d}\alpha + \delta\beta + \gamma$. The Hodge decomposition theorem can be used to show that the equation $\Delta\omega = \lambda$, where ω and λ are p-forms, has a solution if and only if λ is orthogonal to the space H^p of harmonic p-forms; i.e., $\langle\gamma, \lambda\rangle = 0$ for all $\gamma \in H^p$.

5.1 Gauge Invariance

If the domain is simply connected, the integrals of Eq. (5.2) are gauge invariant, as would be expected since the energy must be gauge invariant. This can be seen as follows: (1) Consider first the surface integral. Under the transformation $\boldsymbol{A} \to \boldsymbol{A} + \nabla\chi$ the integral associated with $\nabla\chi$ is,

$$\begin{aligned}\int_S (\boldsymbol{H}\times\nabla\chi)\cdot\mathrm{d}\boldsymbol{S} &= \int_V \nabla\cdot(\boldsymbol{H}\times\nabla\chi)\mathrm{d}V \\ &= \int_V \nabla\cdot(\chi\boldsymbol{J})\mathrm{d}V = \int_S \chi\boldsymbol{J}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0,\end{aligned} \tag{5.20}$$

where $\nabla\cdot\boldsymbol{J} = 0$ has been used. The last equality is true in general if S bounds the currents, and is true for force-free fields in particular since $\boldsymbol{J} = \alpha\boldsymbol{H}$ lies in the magnetic surface S. (2) The integral on the right hand side of Eq. (5.2) is also gauge invariant since under a gauge transformation the integral associated with $\nabla\chi$ is,

$$\int_V \alpha\nabla\chi\cdot\boldsymbol{H}\mathrm{d}V = \int_V \nabla\cdot(\alpha\chi\boldsymbol{H})\mathrm{d}V = \int_S \alpha\chi\boldsymbol{H}\cdot\hat{\mathbf{n}}\mathrm{d}S = 0. \tag{5.16}$$

In a multiply connected domain (for example, the interior of a toroid as discussed in Section 4.2), the terms involving $\boldsymbol{A}$ in Eq. (5.2) are not necessarily gauge invariant since, under a gauge transformation, the function χ may not be single-valued. Yet, for the energy to be well defined in such domains it must be gauge invariant. In Section 4.3, which

discussed gauge invariance and helicity, it was seen that the gauge term corresponded to the additional helicity due to a linking flux. Similarly, the gauge term in the equations for magnetic energy have a physical significance. The physical interpretation of such gauge transformations will be discussed in Section 5.3.1.

5.2 De Rham's Theorems and Homology

The irrotational vector fields in the interior of a three dimensional sphere S can serve as a good example for an introduction to De Rham's theorems. Those irrotational vector fields that can be written as the gradient of a scalar potential field ($\boldsymbol{B} = \nabla\phi$) are known as exact fields. Let $\boldsymbol{r}(t)$ be a smooth curve parameterized by $0 \le t \le 1$. Then the integral of $\boldsymbol{B}$ along $\boldsymbol{r}(t)$ is

$$\int_{\boldsymbol{r}(t)} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{r} = \int_0^1 \left(B_x\frac{\mathrm{d}x(t)}{\mathrm{d}t} + B_y\frac{\mathrm{d}y(t)}{\mathrm{d}t} + B_z\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\mathrm{d}t. \tag{5.22}$$

If $\boldsymbol{r}(t)$ is a closed loop so that $\boldsymbol{r}(0) = \boldsymbol{r}(1)$, the integral on the left hand side of Eq. (5.22) is called the "period" of $\boldsymbol{B}$ around $\boldsymbol{r}(t)$. The terminology comes from the periods of elliptic integrals, and is presumably motivated by Liouville's second theorem. Recall that an elliptic function is a doubly periodic analytic function with no singularities other than poles in the period parallelogram defined by 0, $2\omega_1$, $2\omega_1+2\omega_2$, $2\omega_2$, where $2\omega_1$ and $2\omega_2$ are the two fundamental periods, and the factor of two is conventional. Liouville's second theorem states that the sum of the residues at the poles of an elliptic function in the period parallelogram is equal to zero. This means that for an elliptic function $f(z)$, $(1/2\pi i)\int_C f(z)\,\mathrm{d}z = 0$, the path of integration being the period parallelogram (which may have to be translated so that no pole or zero lies on the boundary of the parallelogram).

It is apparent that if all the periods of $\boldsymbol{B}$ vanish, then $\boldsymbol{B}$ must be irrotational. This is obvious from either Stokes' theorem or from thinking of $\boldsymbol{B}$ as a magnetic field; i.e., if all the periods vanish, one may conclude that no currents are present so that the curl of $\boldsymbol{B}$ must also vanishes. It can also be seen that the following specialized form of the first of De Rham's theorems holds:

I. *An irrotational vector field is exact if and only if all of its periods vanish.*

The proof is simple. Assume that $\boldsymbol{B}$ is exact so that $\boldsymbol{B} = \nabla\phi$. Then

$$\int_{\boldsymbol{r}(t)} \boldsymbol{B}\cdot\mathrm{d}\boldsymbol{r} = \int_0^1 \left(\frac{\partial\phi}{\partial x}\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \frac{\partial\phi}{\partial y}\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \frac{\partial\phi}{\partial z}\frac{\mathrm{d}z(t)}{\mathrm{d}t}\right)\mathrm{d}t. \tag{5.23}$$

The first term in the integral on the right hand side is $\frac{d}{dt}\phi[x(t)]$, and a similar statement can be made for each of the other terms. The integral is then a sum of terms like $\phi[x(t)]|_0^1$. If $x(1) = x(0)$, and the same is true for the y and z terms, the right hand side of Eq. (5.23) vanishes, so that the periods vanish.

Conversely, assume that the periods vanish. Then for any path $\boldsymbol{r}(t)$ between two points p_0 and p

$$\phi(p) = \int_{r(t)} \boldsymbol{B}\cdot d\boldsymbol{r} \tag{5.24}$$

is independent of path. Taking the gradient therefore yields $\boldsymbol{B} = \nabla\phi$ so that $\boldsymbol{B}$ is exact.

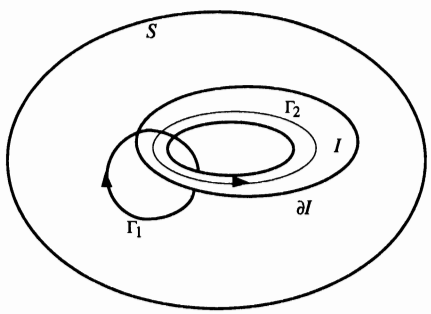

Fig. 5.1. The sphere S contains a current carrying region I which is deleted from the 3-dimensional domain of the interior of S.

To deal with the case where the periods do not vanish, the magnetostatic analogy is again useful. Consider a current loop, which will be defined as a closed tubular neighborhood I that contains the current. The region I containing the current [See Fig. 5.1] is to be thought of as being excised from the 3-dimensional domain of S where the irrotational fields $\boldsymbol{B}$ are defined.

The statement of De Rham's second theorem requires the concept of homology classes which, in the context of closed curves, can be introduced as follows: A closed curve Γ is said to be homologous to zero if it bounds a surface Σ. A circle drawn in the plane is bounded by its edge. However, if the region of the plane bounded by Γ has a hole in it, and the edges corresponding to the annular region are given an orientation as shown in Fig. 5.2, then $\Gamma - \Gamma'$ bounds Σ while Γ alone or $\Gamma + \Gamma'$ do not. Two curves are said to be homologous if their difference bounds; here $\Gamma - \Gamma' \sim 0$ and $\Gamma \sim \Gamma'$.

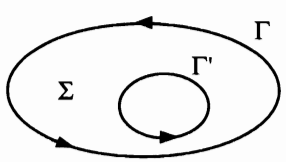

Fig. 5.2. The boundary of the plane region Σ is $\Gamma - \Gamma'$ The curve Γ is homologous to the curve Γ'

The collection of all closed curves form the 1-dimensional cycle group Z_1, and the collection of all boundaries form the 1-dimensional boundary group B_1. The factor group $H_1 = Z_1/B_1$ is the 1-dimensional homology group, and elements of H_1 are the 1-dimensional homology classes.

Referring back to Fig. 5.1, a cycle basis for the homology group $H_1(\partial I)$, where ∂I means the boundary of I, is given by representatives from each of the two homology classes; e.g., by the set of curves $\{\Gamma_1, \Gamma_2\}$, which are, of course, homologous to curves taken on the common boundary. The class of

curves Γ_1 generates H_1 for the interior of the sphere S, while the class of curves Γ_2 generates $H_1(I)$. Thus, the homology group of closed curves on ∂I is the direct sum of the homology groups in I and $S - I$. This is a result of the Alexander duality theorem, which in three dimensions states that the number of independent elements of the n-dimensional homology group in S, where $n = 0, 1, 2$, is equal to that of the $(3-n-1)$-dimensional homology group of $\bar{S}$. $\bar{S}$ is composed of two components: the simply connected space exterior to the sphere S and the doubly connected space interior to I. For $n = 1$, the duality theorem states that the closed curve group in S is dual to the closed curve group in $\bar{S}$. The basis $\{\Gamma_1, \Gamma_2\}$ for the closed curve group on ∂I is then composed of representations from the homology classes of the 1-dimensional homology groups of S and $\bar{S}$.

If Γ_1' is another closed loop in the domain exterior to I, then the homology class of Γ_1' is an integer multiple of Γ_1. Assuming the orientation given by the right hand rule, as shown in the figure, this integer is the linking number of Γ_2 and Γ_1', $\mathbb{L}[\Gamma_1', \Gamma_2]$, discussed in Chapter 4. Note that if Γ_1' does not link Γ_2, the linking number is of course zero.

The periods of $\boldsymbol{B}$ are then

$$\int_{\Gamma_1'} \boldsymbol{B}\cdot d\Gamma_1' = \mathbb{L}[\Gamma_1', \Gamma_2], \tag{5.25}$$

with the obvious notation $d\Gamma_1'$ for an infinitesimal vector displacement along Γ_1'. This is essentially Ampere's circuital law for a unit current. If the linking number vanishes, Γ_1' does not enclose I, and a linking number of n corresponds to going around the circuit n times. This suggests the following specialized form of De Rham's second theorem:

> II. *Let Γ_i be the closed loops which form the basis for H_1. If each period is assigned a number* per (Γ_i), *there is an irrotational vector field which has the assigned periods* $\int_{\Gamma_i} \boldsymbol{B}\cdot d\Gamma_i = \text{per}\,(\Gamma_i)$ *for each* Γ_i.

Before stating the general form of De Rham's theorems, it will be necessary to expand the terminology just introduced.[7] If one generalizes the concept of the triangulation of a surface, discussed in Chapter 4, to higher dimensional spaces where the basic elements of the triangulation are n-simplices, a cycle is defined as a chain whose boundary vanishes, while an n-chain is a formal sum, with constant coefficients, of n-simplices. The collection of all n-cycles forms the n-dimensional cycle group Z_n, while the collection of all n-boundaries forms the n-dimensional boundary group B_n. The factor or quotient group $H_n = Z_n/B_n$ is known as the n-dimensional homology group. The elements of H_n are called homology classes, and cycles in the same homology class are said to be homologous. The H_n can be given a geometrical interpretation: H_0 is a measure of the number of connected components of the space; H_1 of the number of closed curves that are not boundaries of a 2-dimensional part of the space; and H_2 measures the number of closed 2-dimensional surfaces that are not boundaries of some 3-dimensional part of the space.

De Rham's theorems are expressed here in the form given by Flanders.[8] If ω is a closed p-form, for each p-cycle z one can define a period of ω by $\int_z \omega$. The period only depends on the homology class of z. If z is a boundary, then by Stokes' theorem the period vanishes. Therefore, if $\Sigma a_i z_i$ = boundary, then $\sum a_i \int_{z_i} \omega = 0$. De Rham's first and second theorems can then be expressed as:

I. *A closed p-form is exact if and only if all its periods vanish.*

II. *If each p-cycle z is assigned a number* per(z), *there is a closed p-form* ω *which has the assigned periods* $\int_z \omega = \mathrm{per}(z)$ *for each p-cycle z (subject to the consistency relation* $\Sigma a_i \mathrm{per}(z_i) = 0$ *if* $\Sigma a_i z_i$ *is a boundary).*

If ω is set equal to the 2-form B, the substance of De Rham's first theorem (the one that will be most used here) is as follows: since the magnetic field B is solenoidal ($\mathrm{d}B = 0$), a vector potential A exists locally such that $B = \mathrm{d}A$; i.e., B is locally exact. For A to exist globally, the closed surface periods $\int_z B$ must vanish (z being a 2-cycle). The non-existence of magnetic monopoles guarantees that this will always be the case. In general, the domain within which a closed p-form will be exact depends on p: if $p = 0$, the domain must be connected; if $p = 1$, the domain must be simply connected; and if $p = 2$, one must be able to shrink any spherical surface to a point.

Even though the closed surface periods vanish, it is interesting to supplement the solenoidal condition for $\boldsymbol{B}$ with the boundary condition that the normal component of $\boldsymbol{B}$ vanish.[9] While this condition alone would imply that the closed surface periods vanish (even if magnetic monopoles existed), it also allows one to use the concept of relative homology to define *open* surface periods, in terms of the homology modulo the boundary,[10] which can be interpreted as fluxes. If Σ and Σ' are such that $\Sigma' \sim \Sigma$ [mod ∂D], where ∂D is the boundary of the domain D, then $\Sigma - \Sigma'$ can be made into a closed bounding surface by the addition of part of the boundary. The integral $\int_\Sigma \boldsymbol{B} \cdot \hat{\mathrm{N}}\, \mathrm{d}\Sigma = \int_\Sigma B$ is then a period. Interestingly enough, the converse is also true: If the open surface integrals $\int_\Sigma B$ only depend on the homology class (if they are periods), then the closed surface periods of $\boldsymbol{B}$ vanish and the normal component of $\boldsymbol{B}$ also vanishes on the boundary.

It is also possible to define the homology of open arcs modulo the boundary. Two points are said to be homologous if they bound an arc. If these points are on the boundary of D, then two arcs, C and C' contained in D, are said to be homologous modulo the boundary, $C \sim C'$[mod ∂D], if their 0-dimensional boundaries are homologous in $\bar{D}$. Thus, when $C \sim C'$[mod ∂D] it is possible to complete $C - C'$ to a closed, bounding curve in D by adding arcs on the boundary.[11]

In what follows, various integrals involving the scalar or vector potential will be evaluated by introducing cuts on open surfaces Σ or along open arcs.

5.3 Physical Interpretation of the Magnetic Energy Boundary Term

With reference to Fig. 5.3, let the domain D be the exterior of a toroidal surface S_1, bounded by S_0 and S_1. $\bar{D}$ then has two components: the doubly connected toroidal space interior to S_1 and the singly connected space exterior to S_0. Assume that there exists a 1-form X such that $\mathrm{d}X = 0$ and a 2-form Y such that $\mathrm{d}Y = 0$. Then there exist locally a 0-form (function) ϕ such that $X = \mathrm{d}\phi$ and a 1-form A such that $Y = \mathrm{d}A$. ϕ can be made single-valued by taking the open surface Σ as a cut.

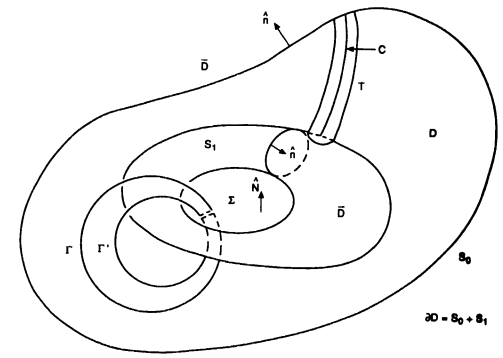

Fig. 5.3. The domain D exterior to a toroidal surface S_1 bounded by S_0 and S_1. The open surface Σ and the open arc C are taken as cuts to make ϕ and A, respectively, single valued.

Consider the integral $\int_D X \wedge Y$. Using the relationship

$$\mathrm{d}(\phi Y) = \mathrm{d}\phi \wedge Y + \phi\, \mathrm{d}Y, \tag{5.26}$$

this integral can be written as,

$$\int_D X \wedge Y = \int_D \mathrm{d}(\phi Y) = \int_{\partial D} \phi Y + \int_\Sigma [\phi] Y, \tag{5.27}$$

where $[\phi] = \int_\Gamma \mathrm{d}\phi = \int_\Gamma X$ is the jump across Σ. This jump is independent of the path Γ chosen provided Γ circles the torus. To see this, note that the two paths Γ and Γ' of Fig. 5.3 can be connected along both sides of the open surface cut Σ (as indicated). If one integrates along the closed path consisting of these connections and Γ and Γ', the contributions of the connections on either side of Σ will cancel. Stokes' theorem then allows the integral over this closed path to be replaced by the integral of $\mathrm{d}X$ over the enclosed surface bounded by $\Gamma - \Gamma'$, which vanishes since $\mathrm{d}X = 0$. Thus, taking account

of the direction of integration along Γ and Γ', the integral over Γ is equal to that over Γ' and $[\phi]$ is independent of the choice of Γ showing that $[\phi]$ is constant on Σ. This means $[\phi]$ can be removed from under the integral and Eq. (5.27) can be written as

$$\int_D X\wedge Y = \int_{\partial D} \phi Y + \int_\Gamma X \int_\Sigma Y. \tag{5.28}$$

Alternatively, one can introduce the open arc C (connecting the component of $\bar{D}$ inside the torus with that outside S_0) as a cut so as to make A single-valued. Then instead of Eq. (5.26), the relation

$$\mathrm{d}(A\wedge X) = \mathrm{d}A\wedge X - A\wedge \mathrm{d}X \tag{5.29}$$

can be used to write,

$$\int_D X\wedge Y = \int_D \mathrm{d}(A\wedge X) \cong \int_{\partial D} A\wedge X + \int_T A\wedge X. \tag{5.30}$$

Here, the arc C has been enclosed in a tube T and the symbol $\cong$ has been used to indicate that small elements of surface, which will vanish as the diameter of the tube is allowed to approach C in the limit, have been ignored. Combining Eqs. (5.28) and (5.30),

$$\int_{\partial D} A\wedge X \cong \int_{\partial D} \phi Y + \int_\Gamma X \int_\Sigma Y - \int_T A\wedge X. \tag{5.31}$$

Evaluating the last integral of the latter expression is most readily accomplished by transforming to classical notation. The integrand $A\wedge X$ becomes $(\boldsymbol{A}\times\boldsymbol{X})\cdot\hat{\mathbf{n}}\mathrm{d}T$, where $\mathrm{d}\boldsymbol{T}$ is the element of area on the tube oriented along the normal $\hat{\mathbf{n}}$.

Noting that the tube is the topological product of a circle, L, (with tangent $\mathrm{d}\boldsymbol{l}$) and the cut, C, (with tangent $\mathrm{d}\boldsymbol{l}'$), the element of surface area can be written as $\mathrm{d}\boldsymbol{T} = \mathrm{d}\boldsymbol{l}\times\mathrm{d}\boldsymbol{l}'$. The integrand is then $(\boldsymbol{A}\times\boldsymbol{X})\cdot(\mathrm{d}\boldsymbol{l}\times\mathrm{d}\boldsymbol{l}')=(\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l})(\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{l}') - (\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}')(\boldsymbol{X}\cdot\mathrm{d}\boldsymbol{l})$. The space $\bar{D}$, interior to the toroid, is singly connected by virtue of the cut S, while the cut C connects the two components of $\bar{D}$. Then, by Stokes' theorem, the last term can be seen to vanish since $\nabla\times\boldsymbol{X}=0$. Consider the remaining two terms in the limit that $T\to C$. Stokes' theorem can again be used to transform the $\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{l}$ term; and, because $\boldsymbol{X}$ approaches a value depending only on its limiting position on C, the integral can be written as the product

$$\int_T (\boldsymbol{A}\times\boldsymbol{X})\cdot\hat{\mathbf{n}}\,\mathrm{d}T = \int_{\partial D} \boldsymbol{Y}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \int_C \boldsymbol{X}\cdot\mathrm{d}\boldsymbol{l}'. \tag{5.32}$$

Thus, Eq. (5.31) can be written as

$$\int_{\partial D} A\wedge X = \int_{\partial D} \phi Y + \int_\Gamma X \int_\Sigma Y - \int_{\partial D} Y \int_C X. \tag{5.33}$$

The general form of Eq. (5.33), not yet written in terms of magnetic fields and potentials, can now be used in the case that ∂D is a magnetic surface. Let A now be identified with the vector potential 1-form. If the current vanishes outside the torus, an appropriate choice for X and Y is $X=*B$ and $Y=B$. These are consistent choices since $\mathrm{d}X = \mathrm{d}(*B)=\mu_0 *\mathrm{J}=0$ and $\mathrm{d}Y=0$ in D. With the condition that the normal component of $\boldsymbol{B}$ vanish on ∂D, the open surface integral $\int_\Sigma B$ is a period which can be interpreted as a flux, and Eq. (5.33) becomes

$$\int_{\partial D} A\wedge *B = \int_\Gamma *B \int_\Sigma B = \mu_0 I_\phi \Phi. \tag{5.34}$$

where I_ϕ is the azimuthal current inside the torus (outside the domain D) and Φ is the flux through the surface Σ. Since the current vanishes in the domain D, Stokes' theorem implies that

$$\int_{\partial D} A\wedge *B = \int_D B\wedge *B = \langle B,B\rangle. \tag{5.35}$$

Eq. (5.34) can be used to interpret the boundary term of Eq. (5.17) in the case that the volume of integration is the interior of a torus. Attention, however, must be paid to the orientation of the boundary surface. To arrive at Eqs. (5.34) and (5.35), the domain D was chosen to be the exterior of the toroidal surface; on the other hand, the volume of integration in Eq. (5.17) is now assumed to be the interior of the torus. A surface element on the boundary thus has opposite orientation for the two cases, which will introduce a minus sign when substituting the value of the integral into Eq. (5.17).

Thus, not surprisingly, the total energy as expressed by Eq. (5.17) is the sum of the energy interior to the torus, where currents may be present, and that in the field exterior to the torus, represented by the boundary term. Since it has been assumed that there are no currents in the domain exterior to the torus, the field in this region must be due to currents within the torus. Currents outside S_0 do not contribute to the magnetic energy in the space

bounded by S_0+S_1 because of the boundary conditions imposed and the fact that the space outside S_0 is simply connected. If the boundary configuration were different, so that the space outside S_0 was not simply connected (for example if S_0 was a second torus so that S_1 was nested within it), then there could be additional contributions to the field energy between the toroids due to currents exterior to S_0.

5.3.1. Physical Examples

The role of multi-valued gauge transformations will be discussed in terms of two physical examples: a simple ring current and a force-free configuration, both in a toroidal domain. It will be seen that gauge transformations in multiply connected domains can have real physical effects. An example of this, which will not be treated here, is given by the Aharonov and Bohm effect. This has been discussed in the context of universal covering spaces (touched on briefly below) by Schulman.[12]

5.3.1.1. Ring Current in a Toroidal Domain

Consider a current I_ϕ in a perfectly conducting toroidal domain D. The energy of the configuration can be written in terms of the currents in the domain as

$$E = \frac{1}{2}\int_D A\wedge *J. \tag{5.36}$$

For consistency, it is necessary that $\int_{\Gamma_1'} A = \Phi_{\Sigma_2}$ [See Fig. 5.4(b)]. The meaning of a gauge transformation $A'=A+d\chi$ with χ a multivalued function will be discussed shortly. Using Stokes' theorem, Eq. (5.36) becomes

$$\mu_0 E = \frac{1}{2}\int_D B\wedge *B - \frac{1}{2}\int_{\partial D} A\wedge *B\ . \tag{5.37}$$

The first term is clearly the energy of the field within the toroidal conductor, while Eq. (5.34) and the discussion following Eq. (5.35) allow the boundary term to be written as

$$-\int_{\partial D} A\wedge *B = \int_{\Gamma_2} *B\int_{\Sigma_2} B = \mu_0 I_\phi \Phi_{\Sigma_2}. \tag{5.38}$$

To see this, note that the first integral in the product, using Stokes' theorem, is

$$\int_{\Gamma_2} *B = \int_{\Sigma_1} d(*B) = \mu_0\int_{\Sigma_1} *J = \mu_0 I_\phi. \tag{5.39}$$

Thus, the energy is the sum of the field energy within the current carrying toroidal domain and the energy in the field exterior to the domain due to currents within D, the latter energy being represented by the boundary term.

A "multi-turn coil" having n "turns" would have Γ_1' traversed n times. This is equivalent to performing a gauge transformation $A'=A+d\chi$ in Eq. (5.36). This can be seen by considering the $d\chi$ term, $\int_D d\chi\wedge *J$, in such a transformation. Because $d\chi$ is a closed 1-form, and $*J$ a closed 2-form, the discussion preceding Eq. (5.28) allows this term to be written as $\int_{\Gamma_1} d\chi\int_{\Sigma_1} *J = I_\phi\int_{\Gamma_1'} d\chi$. (The fact that $\int_{\partial D}\chi(*J) = 0$ has been used). Now the jump in the function χ across Σ_1, $[\chi] = \int_{\Gamma_1'} d\chi$, must yield Φ_{Σ_2} for each "turn" of the coil. Thus, the energy in the field exterior to the toroid would be $n I_\phi \Phi_{\Sigma_2}$. Thus, for this example, a gauge transformation corresponds to multiplying the energy in the field by some integer n.

If a multiply connected space is *locally* simply connected, it is possible to define a universal covering space that is simply connected. A space is locally simply connected if every point has at least one simply connected neighborhood. The universal covering space is the largest covering of the multiply connected space. The existence of a universal covering space allows the properties of a multiply connected space to be studied in an analogous space that is simply connected. One of the best examples of a covering space is that for a 1-dimensional torus; that is, a circle. The 1-torus, or for that matter the n-torus, is not simply connected. The covering space defined by a helix, which when projected onto the plane perpendicular to its axis is the circle, is simply connected. In general, if f is a continuous mapping of a covering space onto a topological space, then f^{-1} is a multivalued homeomorphism (where the images and inverse images of open sets are themselves open). A covering space is a universal covering if it is simply connected.

Returning to the example above, if one constructs the simply connected, universal covering space $\mathbb{D}$ of the domain D interior to the torus, n is essentially the winding number, uniquely determined by the path homotopy class of Γ_1'. In general, gauge transformations in $\mathbb{D}$ will affect the physics in D.

Note that because the normal component of $\boldsymbol{B}$ vanishes on ∂D, $\int_{\Sigma_2} B$ is an open surface period. The second of De Rham's theorems then guarantees the existence of a B exterior to D such that

$$\int_{\Sigma_2} B = \int_{\Gamma_1'} A = \Phi_{\Sigma_2}. \tag{5.40}$$

However, while the existence of B is assured, its uniqueness is not; in this connection see the comments just before the beginning of Section 5.3.1.

5.3.1.2. Application to Force-Free Fields

Perhaps the best known solution to the cylindrically symmetric, force-free magnetic field equations with constant α is the Lundquist[13] solution given by

$$\boldsymbol{H} = A_0 J_1(\alpha r)\hat{\phi} + A_0 J_0(\alpha r)\hat{\boldsymbol{z}}, \tag{5.41}$$

where A_0 is an arbitrary constant.

If one chooses to apply this solution in a cylindrical region of radius a, such that $J_0(\alpha a) = 0$, this solution matches smoothly -- no surface currents are required -- to an external field given by $[0, (aA_0/r)J_1(\alpha a), 0]$, which only has a non-vanishing $\hat{\phi}$ component. Since there is a non-vanishing field outside the cylindrical boundary at $r=a$, it is clear that the surface integral discussed in the last section cannot vanish. Nevertheless, consider the following *incorrect* argument: Since α is a constant, $\nabla\times\boldsymbol{B}=\alpha\boldsymbol{B}$ can be written in terms of $\boldsymbol{A}$ as $\nabla\times(\nabla\times\boldsymbol{A} - \alpha\boldsymbol{A}) = 0$. This means that $\boldsymbol{B} = \alpha\boldsymbol{A}$ and the surface integral vanishes since $\boldsymbol{H}\times\boldsymbol{A} = 0$.

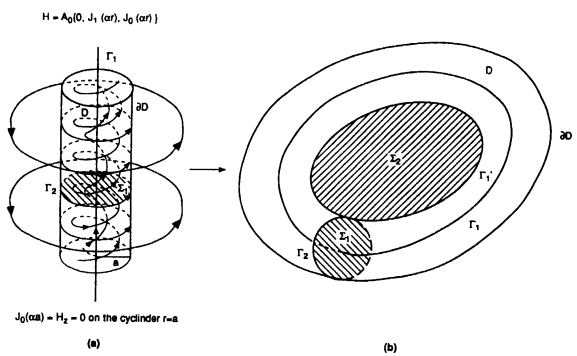

Fig. 5.4. The Lundquist solution can be given the topology of a torus by identifying $z = \pm\infty$ (finite integrals on Γ_1 and Σ_2 can be had by identifying $z = \pm l$ rather than $z = \pm\infty$). The domain D where the force-free condition holds in interior to the torus.

Why is this an incorrect argument? There are two reasons: (1) unless $z=-\infty$ and $z=+\infty$ are identified, there must be additional return currents outside the cylindrical domain. (2) Assume that henceforth this identification is made; then the issue of return currents is eliminated, but the solution will have the topology of a torus. Since the equation $\nabla\times(\nabla\times\boldsymbol{A} - \alpha\boldsymbol{A}) = 0$ actually implies that $\boldsymbol{B} = \alpha\boldsymbol{A} + \nabla\chi$, the statement that $\boldsymbol{H}\times\boldsymbol{A} = 0$ is not gauge invariant because χ is a multivalued function in such a multiply connected domain.[14]

While identifying $z = -\infty$ and $z = +\infty$ gives the topology of a torus, it is essentially one with an infinite principal radius. To obtain finite integrals, $z = \pm l$ should be identified rather than $z = \pm\infty$. While the Lundquist solution can be matched to an external field

without surface currents, matching the internal field of a finite torus to one exterior to the torus requires an azimuthal surface current, as discussed by Buck.[15]

The force-free relation $d(*B)=\alpha dA$, with α being a constant, implies that $*B=\alpha A+d\chi$. The boundary term is then given by (see Figure 5.4),

$$\int_{\partial D} A\wedge *B = \int_{\partial D} A\wedge d\chi = \int_D d(\chi B) = \int_{\partial D} \chi B + \int_{\Sigma_1} [\chi]B \tag{5.42}$$

where the fact that $d(A\wedge d\chi) = B\wedge d\chi = d(\chi B)$ has been used. The first term on the right hand side of Eq. (5.42) vanishes and, by the argument following Eq. (5.27), $[\chi]$ can be taken out of the second integral. Thus,

$$\int_{\partial D} A\wedge *B = \int_{\Gamma_1} d\chi \int_{\Sigma_1} B. \tag{5.43}$$

Using the force-free relation, the integral over Σ_1 becomes

$$\int_{\Sigma_1} B = \frac{1}{\alpha}\int_{\Sigma_1} d(*B) = \mu_0 \frac{I_\phi}{\alpha}, \tag{5.44}$$

where I_ϕ is the azimuthal current. Since $[\chi]$ is constant on Σ_1, Γ_1 may be chosen in ∂D, bounding Σ_2 (shown as Γ_1' in Fig. 5.4). Choose A' such that $\int_{\Gamma_1'} A' = \Phi_{\Sigma_2}$. Then in the interior domain D, since $*B=\alpha A+d\chi$, the objective will be to find $d\chi$ such that $\int_{\Gamma_1'} A = \int_{\Gamma_1'} A'$. Now, $A = \frac{1}{\alpha}(*B-d\chi)$, so that

$$\int_{\Gamma_1'} A = \frac{1}{\alpha}\int_{\Gamma_1'} *B - \frac{1}{\alpha}\int_{\Gamma_1'} d\chi. \tag{5.45}$$

The first term on the right hand side vanishes since $\int_{\Gamma_1'} *B = \mu_0\int_{\Sigma_2} *J = 0$. Thus, the requirement that $\int_{\Gamma_1'} A = \int_{\Gamma_1'} A'$ implies, with the choice $\chi\to-\chi$, that

$$\int_{\Gamma_1'} d\chi = \alpha\Phi_{\Sigma_2}. \tag{5.46}$$

Equations (5.44) and (5.46) then give for the boundary term,

$$\int_{\partial D} A \wedge * B = \mu_0 \Phi_{\Sigma_2} I_\phi, \tag{5.47}$$

which will vanish only if the flux through Σ_2 vanishes; i.e., I_ϕ=0. Thus, the interpretation of this term for force-free fields is consistent with that given in the previous section: it represents the energy in the field exterior to ∂D.

If, for some region of a given surface, the normal component of $\boldsymbol{B}$ vanishes, the vector potential $\boldsymbol{A}$ can be written in terms of a "local surface gradient." That is, let the normal to the surface be oriented along the $\hat{z}$ direction. Since the normal component of $\boldsymbol{B}$ vanishes, $B_z = \partial_x A_y - \partial_y A_x = 0$, which means that $A_y = \partial_y \chi$ and $A_x = \partial_x \chi$. The 1-form A can therefore be represented as a local surface gradient in the sense that $A = \mathrm{d}_s \chi = \partial_x \chi \mathrm{d}x + \partial_y \chi \mathrm{d}y$.

For the Lundquist solution, the component of $\boldsymbol{B}$ normal to ∂D vanishes, implying that A is a local surface gradient. Since A is a closed 1-form on ∂D, by De Rham's first theorem, A will be exact (χ will be single-valued) if $\int_{z_i} A = 0$, where z_i are the independent 1-cycles on ∂D; i.e., if $\int_{\Gamma_1} A = \int_{\Gamma_2} A = 0$, or $\int_{\Sigma_2} B = \int_{\Sigma_1} B = 0$. This condition is equivalent to I_ϕ=0, and can be met with a non-vanishing field in the interior of the torus if αr_0, where r_0 is the minor radius of the torus, is a zero of $J_1(\alpha r)$. This can be seen from,

$$I_\phi = 2\pi \frac{A_0}{\alpha} \int_0^{\rho_0} J_0(\rho)\rho \, \mathrm{d}\rho = 2\pi \frac{A_0}{\alpha} \rho_0 J_1(\rho_0), \qquad \rho = \alpha r \tag{5.48}$$

or, by simply noting that the $\hat{\phi}$ component of the field given by Eq. (5.41) has a zero at $J_1(\alpha \mathrm{r}) = 0$, and that by Ampere's circuital law, the current density integrated over the enclosed area must vanish.

References

[1]Note that while this result is consistent with Yang (Reference 2), it differs from the expression obtained by Reiman (Reference 5) who *defines* the energy as $\frac{1}{2}\int_V \boldsymbol{B}^2$. The definition of magnetic energy given here is consistent with, for example, J. R. Reitz and F.J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA., 1960).

[2]W.H. Yang, *Astrophys. J.* **348**, L73 (1990).

[3]L. Woltjer, *Proc. Nat. Acad. Sci. U.S.A.* **44**, 489 (1958).

[4]H.K. Moffatt, *J. Fluid Mech.* **159**, 359 (1985).

[5]A. Reiman, *Phys. Fluids* **23**, 230 (1980).

[6]G.F.D. Duff and D.C. Spencer, *Ann. of Math.* **56**, 128 (1952).

[7]J.H. Fraleigh, *A First Course in Abstract Algebra* (Addison-Wesley Publishing Co., Reading, Massachusetts, 1968).

[8]H. Flanders, *Differential Forms* (Academic Press, New York, 1963).

[9]In a simply connected domain, where the current vanishes so that $\boldsymbol{B}$ is harmonic ($\nabla^2 \boldsymbol{B} = 0$), if the normal component of $\boldsymbol{B}$ vanishes on the boundary, $\boldsymbol{B}$ will be identically zero.

[10]A.A. Blank, K.O. Friedrichs, and H. Grad, *Theory of Maxwell's Equations Without Displacement Current*, Notes on Magneto-Hydrodynamics V, NYO-6486 (1957).

[11]These results are related to the Alexander duality theorem discussed in Reference 10 and, for example, J.R. Munkres, *Elements of Algebraic Topology* (Addison-Wesley, Inc., New York, 1984).

[12]L.S. Schulman, *J. Math. Phys.* **12**, 304 (1971).

[13]S. Lundquist, *Arkiv för Fysik* **2**, 361 (1951).

[14]If $z = -\infty$ and $z = +\infty$ are not identified, the following argument might seem attractive: The $\nabla \cdot \boldsymbol{B} = 0$ implies that $\nabla^2 \chi + \alpha \nabla \cdot \boldsymbol{A} = 0$. If the gauge $\nabla \cdot \boldsymbol{A} = 0$ is chosen so that $\nabla^2 \chi = 0$, since χ is harmonic throughout the domain, it is identically zero. This argument is, however, also incorrect since the domain is restricted to the interior of the cylinder.

[15]G.J. Buck, *J. Appl. Phys.* **36**, 2231 (1965).

Chapter 6

APPLICATIONS

Force-free magnetic fields play an important role in astrophysics and plasma physics, and possibly in type-II superconductors carrying a current parallel to the flux line lattice. From an experimental perspective, coils have also been wound for high-field electromagnets that approximate force-free configurations so as to reduce the stress to which the windings are subjected. Such force-reduced magnet designs are potentially useful in thermonuclear and energy storage applications where high magnetic fields are needed.

As was discussed in Chapter 2, finite force-free magnetic field configurations are subject to the limitations set by the virial theorem, which relates the stored magnetic energy of the system to the integral of the trace of the stress tensor over the magnetic field volume. While stresses may be eliminated in a given region, they cannot be canceled everywhere. In general, the use of force-free field configurations allow the forces needed to balance the outward pressure of the magnetic field to be reduced in magnitude by spreading them out over a larger region. While the virial theorem does indeed set limits for the minimum amount of structure required for magnets made of materials having a given strength (see Section 2.2), force-free configurations nonetheless appear to have the potential to contribute to improved designs for large magnets. It is, for example, possible to design a truly force-free coil surrounded by a force-bearing ring.[1] From a practical point of view, since structural material is inexpensive compared to complex superconducting coils, force-reduced configurations are only likely to be employed for very special applications. Hill, et al.[2] have shown that it may be possible to wind hybrid coils that are force-reduced

only in a particular region of interest. His results could significantly reduce the cost and widen the area of applicability of such coils.

The discovery of high-temperature superconductors has also renewed interest in force-reduced magnetic field configurations. Because the materials that exhibit superconductivity at high temperature are ceramics, and tend to be quite brittle, it is important to reduce the forces they might experience if they are to be used for high field applications. In addition, Furth[3] has raised the possibility that force-free field configurations may have the potential to raise the critical magnetic field and current-density limits in such superconductors.

One of the most important applications of force-free fields in astrophysics is to solar prominence models. Because of this, and the fact that the applications to electromagnets and plasma physics are relatively mature, this chapter will limit its discussion to the role of force-free fields in solar prominence models and the theoretical conjecture that force-free fields may play a role in type-II superconductors. Each of these areas has an extensive history, and the cited references should serve as an introduction to the literature. The treatment of solar prominences will concentrate on the storage of energy in twisted magnetic fields.

6.1 Solar Prominence Models

The equation of motion for a plasma under the influence of a pressure gradient, a magnetic field, and gravity is

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -\nabla p + \boldsymbol{j}\times\boldsymbol{B} + \rho \boldsymbol{g}. \tag{6.1}$$

If the flow speed $\boldsymbol{v}$ is much smaller than both the isothermal velocity of sound and the Alfvén speed, Eq. (6.1) reduces to the equation of magnetohydrostatics

$$\nabla p = \boldsymbol{j}\times\boldsymbol{B} + \rho \boldsymbol{g}, \tag{6.2}$$

which in principle can be used to describe the static, preflare configuration of a solar flare.[4] When the ratio of the plasma pressure to magnetic field pressure, known as the plasma β, is negligible compared to unity, and the height of the preflare configuration is much smaller than the scale height so that the gravity term can be neglected, the field is approximately force-free.

Consider a flux tube emerging from the sun's photosphere into the chromosphere to form an arch. In the photosphere or below, material motions will dominate and the magnetic field structure will be assumed to be unknown; in the chromosphere, and above in the corona, the gas pressure is small and the field must be approximately force-free. Note that, with regard to terminology, it is common to consider potential fields, where the current vanishes, to be force-free with vanishing α; linear force-free fields to be those with α equal to a constant; and non-linear force-free fields to be those having α a function of position.

The magnetic field associated with a solar prominence is often modeled as a twisted flux tube. Twisting can be introduced either by photospheric twisting motions (due to Coriolis forces) at the locations where the base of the arch enters the photosphere,[5] or by flux cancellation; i.e., by shear and convergence along the neutral line separating two regions of opposite magnetic polarity, followed by reconnection of the field lines.[6] In either case, currents are introduced that flow along the field lines in the force-free region. While the former case is easily visualized, the latter is shown in Fig. 6.1. Here the original potential field bridges the neutral line. Subsequently, magnetic shear is produced by photospheric flows both along the neutral line as well as toward the neutral line. This model assumes that the small, high curvature loop formed during the reconnection of the magnetic field lines submerges into the photosphere leaving the twisted field which has a larger radius of curvature. The latter is sustained above the photosphere by magnetic buoyancy forces. As either the twisting or shear and convergence mechanisms increases the magnetic flux in the twisted field, the height of the configuration above the photosphere increases. If relatively cool plasma resides in the magnetic configuration, the structure forms a prominence.

The type of twisted field that ultimately results from either twisting mechanism is crucial to the discussion of the magnetic energy. Parker[7] has forcefully argued that one cannot expect to find in nature flux ropes with a structure more complex than a more or less uniform twist across the radius of the tube. Assume then, for purposes of discussion, that the force-free, uniformly twisted solution given in Chapter 4 by Eqs. (4.21) and (4.22) is an appropriate model for the magnetic field associated with a solar prominence.

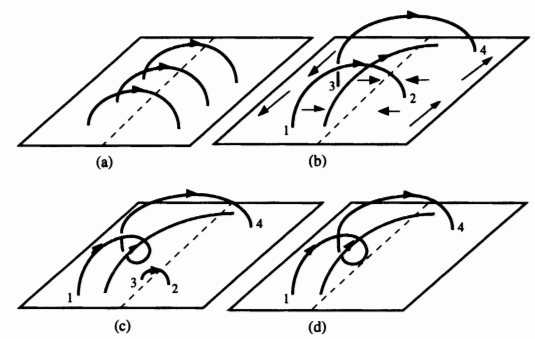

Fig. 6.1. Formation of a twisted magnetic field by flux cancellation. (a) shows the original potential field; (b) the sheared field with photospheric shear and convergence flows; (c) the reconnected field; and (d), the resulting twisted field after submergence of the high curvature flux loop.

The field in the flux tube is confined by a pressure differential across its boundary. Since the flux along the axis of the tube is conserved, the decrease in the mean longitudinal field as the tube is twisted leads to an increase in the radius of the tube. Physically, for an axially symmetric solution such as that given by Eqs. (4.21) and (4.22) to be a reasonable approximation to the twisted flux tube used to model the field associated with a prominence, the aspect ratio (the ratio of the principal to minor radius) of the prominence must be large. Note also that the use of this solution to form a flux tube of finite radius R means that there will be an additional surface current needed to match to any potential field that may be present for $r > R$.

First, consider the helicity in the prominence. Since the field configuration below the photosphere is assumed to be unknown, it is clear that only relative helicities in the sense of Berger and Field,[8] where the field below the photosphere is assumed to be the same for different field configurations in the force-free region above the photosphere, can be computed. For the uniformly twisted flux tube, it was shown in Chapter 4 that $\boldsymbol{A}\cdot\boldsymbol{B}$ vanishes in the volume of integration above the photosphere (the flux tube). Consider now the relative helicity between a uniformly twisted flux tube and a potential field (consistent with the approximation made here, simply a field along the axis of the tube). By symmetry, $\boldsymbol{A}\cdot\boldsymbol{B}$ also vanishes for this field. It is clear from the discussion of Chapter 4 that the non-vanishing of the helicity depends on a gauge term and the fact that the space consisting of the interior of the flux tube and the region below the photosphere is multiply connected.

Since the fields of the twisted flux tube and the untwisted tube are assumed to be the same in the region below the photosphere, the vector potential for each can differ at most by a gauge term $\nabla\chi$. This term can also be allowed to absorb any gauge freedom in the region above the photosphere. Thus the relative helicity reduces to the right hand side of Eq. (4.17) and is consequently given by

$$\Delta\mathcal{H} = \Phi_{\Sigma_1}\Phi_{\Sigma_2}. \tag{6.3}$$

Here Φ_{Σ_1} is the (conserved) flux within the tube and Φ_{Σ_2} is the flux through the surface Σ_2 bounded by the arch and the photosphere (Fig. 4.7 is applicable).

The source of energy for a solar flare is believed to be magnetic in origin, and derived from non-potential twisted fields. Assume again that the magnetic field of a pre-flare solar prominence can be modeled as a flux tube that is uniformly twisted and force-free. The solution of Eqs. (4.21) and (4.22) again give an interesting example. Note that knowledge of the field configuration in the force-free region is insufficient to determine the field energy. This follows from the fact that $\boldsymbol{H}$ in the force-free relation is indeterminate up to an arbitrary multiplicative constant. It is, however, possible to compute relative energies. For example, one might be interested in the energy difference between a twisted arch and a similar arch described by a potential field.

Consider the energy of a uniform field B_0 in an untwisted cylindrical flux tube of radius R_0. As the tube is twisted the radius increases to R. Ignoring for the moment any contribution from surface terms, the magnetic energy in the tube is

$$E = \frac{1}{\mu_0}\int_V (B_z^2 + B_\phi^2)\,\mathrm{d}V = \frac{B_0^2\pi z}{\mu_0}\int_0^R \frac{2r\,\mathrm{d}r}{(1+r^2/a^2)} = \frac{B_0^2}{\mu_0}\pi z a^2\ln(1+R^2/a^2). \tag{6.4}$$

On the other hand, the axial flux in the tube is given by

$$\Phi = 2\pi B_0\int_0^R \frac{r\,\mathrm{d}r}{(1+r^2/a^2)} = B_0\pi a^2\ln(1+R^2/a^2). \tag{6.5}$$

The energy per unit length of the tube is then $(1/\mu_0)B_0\Phi$. Since the axial flux is conserved, and equal to the original flux $B_0\pi R_0^2$, the energy contained in the volume of the flux tube is independent of the twist. This does not, however, mean that energy cannot be stored in uniformly twisted magnetic fields. Twisting the field introduces an axial current density given by $J_z = \alpha H_z$ since the field is force-free. Uniformly twisting a flux tube then results in a current along the tube that creates a magnetic field that links the tube. Using the expressions given by Eqs. (4.21) and (4.22) for α and H_z, the current is

$$I = \frac{4\pi B_0}{a\mu_0}\int_0^R \frac{r\,\mathrm{d}r}{(1+r^2/a^2)^2} = \frac{2\pi B_0 a}{\mu_0}\left[1-(1+R^2/a^2)^{-1}\right]. \tag{6.6}$$

Interestingly enough, there is a value of a that maximizes the current. The conservation of flux implies that,

$$(1+R^2/a^2) = e^{R_0^2/a^2}. \tag{6.7}$$

The current can then be written as,

$$I = \frac{2\pi B_0 a}{\mu_0}(1-e^{-R_0^2/a^2}). \tag{6.8}$$

Differentiating to find the extremum of the current I with respect to a results in the transcendental equation,

$$1 - e^{-1/u^2}(1+2/u^2) = 0. \tag{6.9}$$

where $u = a/R_0$. This can be solved numerically to yield $u = 0.89$. Using this value in the second derivative of the current with respect to a confirms that the extremum is a maximum. The value obtained for u falls into the stable range given by Parker[9] as $u \geq 0.5956$.

In writing Eq. (6.4), the surface term was ignored. In general, the expression for the energy is given by

$$E = \frac{1}{2}\int_V \boldsymbol{H}\cdot(\nabla\times\boldsymbol{A})\,\mathrm{d}V + \frac{1}{2}\int_{\partial V}(\boldsymbol{H}\times\boldsymbol{A})\cdot\mathrm{d}\boldsymbol{S}. \tag{6.10}$$

As has been shown above, the energy contained in the volume of the flux tube is independent of twist. The first term on the right hand side of this equation will therefore not contribute when computing the difference in energy between a flux tube containing a potential field and one having a uniform twist. For a simply-connected domain, the integrals in Eq. (6.10) are gauge invariant. When the domain is multiply connected, the gauge term takes on a physical significance, as discussed in Chapter 5. The surface term does not vanish for the twisted tube, but represents the energy in the field exterior to the domain of integration due to currents within the domain. It yields the value $(1/2)I\Phi_{\Sigma_2}$ where I is given by Eq. (6.6) and Φ_{Σ_2} is again the flux through Σ_2.

As was the case for helicity, introducing a uniform twist into a flux tube does not increase the magnetic energy within the flux tube itself, but creates a current which acts as the source for a field that links the tube. The ambiguity as to whether the energy resides in the current or the field is an old one in magnetostatics.

To make realistic estimates of the energy available from a twisted flux tube for a solar flare, one must address the issue of the post-flare magnetic configuration. If it is assumed that total helicity is conserved, it may well be that a linear force-free field, rather than a potential field, represents the post-flare configuration of the flux tube. During the relaxation process, the discussion given in Chapter 4 on the exchange of link, kink, and twist helicity is relevant to the magnetic reconnection problem. Estimates of the energy available in terms of the topological complexity of the field have been made by Berger.[10]

The argument that the post-flare configuration should be a linear force-free field is based on the work of Woltjer and requires that the Taylor conjecture be true. This was discussed in Chapter 1. The key point is that in deriving the result that a linear force-free field is the lowest energy state that a magnetic field can reach when helicity is conserved, Woltjer used the approximation of ideal magnetohydrodynamics. But this means that the constant α or linear force-free state is topologically inaccessible from most initial configurations. While Taylor's conjecture, that global helicity is conserved while finite diffusivity effects are invoked to allow the field to relax to a linear one, gives one way out of this conundrum, it is not entirely satisfactory from a theoretical point of view.

6.2 Type II Superconductors

When a superconductor is placed in a magnetic field, the screening currents that cancel the flux inside the material flow within a thin surface layer. Consequently, the magnetic flux decays exponentially within the region where the currents are flowing. The penetration depth, λ, of the magnetic field is defined as

$$\lambda = \frac{1}{B(0)} \int_0^\infty B(x)\, dx, \tag{6.11}$$

where x is normal to the superconducting surface and $B(0)$ is the flux density at the surface. Within the superconductor the flux falls off as

$$B(x) = B(0)\, e^{-x/\lambda}. \tag{6.12}$$

As the flux density decays with increasing distance into a superconductor, the number of superconducting electrons increases. The distance over which this rise in superconducting electrons occurs is known as the coherence length ξ. When the coherence length is greater than the penetration depth there is an increase in the total free energy close to the boundary between a normal and superconducting region. Materials displaying this behavior have a positive surface energy and are known as type-I superconductors. Conversely, when the penetration depth is greater than the coherence length, the surface energy is negative and such materials are known as type-II superconductors.

It is energetically favorable for type-II materials, which have a negative surface energy between normal and superconducting regions, to enter a mixed state when an external magnetic field is applied. When the field is normal to a thin sample, the mixed state consists of a triangular close packed lattice (known as the fluxon or flux line lattice) of normal cores parallel to the applied field which are threaded by magnetic flux. Each core has a circulating supercurrent vortex that generates one fluxon ($\Phi_0 = h/2e$) of magnetic flux. A model of these vortices has been given by De Gennes and Matricon.[11]

When a current is passed through a type-II superconducting sample having a transverse applied magnetic field, there is a critical current beyond which a voltage is detected along the sample. The presence of the voltage indicates the onset of flux flow which occurs when the Lorentz force on the flux lattice due to the transport current exceeds the pinning force of the bulk material on the flux line lattice. The configuration that will be discussed here is that of longitudinal geometry. In this case a transport current flows along a type-II superconductor in the presence of a parallel magnetic field. Experimentally, the configuration is often a superconducting cylinder carrying an axial current with a parallel external field. Flux-flow resistance is also detected in the longitudinal case, and because of the

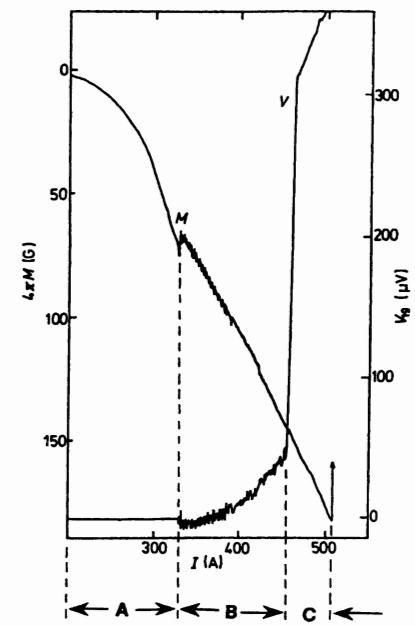

Fig. 6.2. (A) is a non-linear region of increasing paramagnetic moment; (B) is a linear region of increasing paramagnetic moment where flux flow is present; and (C) is a second linear region of increasing paramagnetic moment where large-scale flux flow occurs. [Shown superimposed on data taken from D. G. Walmsley and W.E. Timms, *J. Phys. F: Metal Phys.* **7**, 2373 (1977)].

geometry, its presence indicates that an azimuthal component of flux must be moving inward toward the center of the cylinder in order to generate the observed electric field. An example of the type of experimental data that is obtained in the longitudinal case is shown in Fig. 6.2.

The three fundamental flux regimes illustrated in this figure will be taken as typical for the purposes of discussion. They are defined, as a function of the transport current, as: (A), a non-linear region of increasing paramagnetic moment; (B), a linear region of increasing paramagnetic moment where flux flow is present; and (C), a second linear region of increasing paramagnetic moment where large-scale flux flow occurs. The regions shown in Fig. 6.2 are superimposed on data abstracted from Walmsley and Timms.[12]

For the case of longitudinal geometry, it is usually assumed that the vortices form a helical array so as to account for the observed paramagnetic moment. It is often further assumed, particularly before flux-flow voltages are observed, that the array takes a force-free or nearly force-free configuration following the suggestion of Bergeron.[13] Other theoretical models have been discussed in the reviews by Campbell and Evetts[14] and Timms and Wallmsley.[15] In the case of a transverse field, the model often used tacitly assumes that current flows *throughout the whole body of the metal* and that this transport current interacts via the Lorentz force with the magnetic flux associated with the normal cores of the vortices.[16] If this model is extended to encompass longitudinal geometry, the Lorentz force on the vortices would only vanish if the vortices were parallel to the symmetry axis, and it would not be possible to account for the observed paramagnetic moment.

If, on the other hand, one assumes that the current flows *along* the vortices, it is possible to argue that the lattice assumes a force-free configuration, and that the transport current could be carried without pinning while producing a paramagnetic moment. However, for this to be true, either the density of vortices, the longitudinal current carried per vortex, or both would have to vary so as to satisfy the force-free relation. While some experimental results are not inconsistent with the assumption that the flux-line lattice adopts a force-free configuration,[17] there seems to be no *compelling* experimental or theoretical reason for this to be true. A number of alternative models have been given by Timms and Walmsley.[18]

Assume that the current flows along the vortices comprising the flux line lattice. First, the non-linear increase of the paramagnetic moment with increasing current, as shown in region A of Fig. 6.2, implies that the angle the helical flux-line lattice makes with the symmetry axis must, on the average, be increasing. That is, the lattice must alter its geometric configuration so as to give rise to a greater current-induced magnetization than would result from only an incremental increase in current. This must be true regardless of the model used for the flux-line lattice.

Region B is where the onset of flux-flow resistance is first observed. The linear increase in paramagnetic moment with increasing current implies that the angle the helical flux-line lattice makes with the symmetry axis must, on the average, be constant. This does not mean that the lattice cannot change its configuration, but only that any change must average out over the lattice so there is no net additional geometrical contribution to the paramagnetic moment.

Region C is characterized by the large step increase in the voltage associated with flux flow. Such a large increase indicates that there must be a new flux-flow mode for this region. The linear increase in paramagnetic moment again implies that, on the average, the angle the flux-line lattice makes with the symmetry axis must remain constant.

There is an extensive literature that attempts to explain the experimental results described above, and a number of models have been proposed that would account for the observations, including one by this author.[19] The interested reader is directed to the latter paper and the references cited therein.

The solution to the Ginzburg-Landau equations found by Abrikosov[20] has been used extensively to study the properties of the flux lattice of type-II superconductors when the transport current and an applied magnetic field are directed perpendicular to each other. If it is assumed, in the longitudinal configuration, that current flows along the vortices, the question must be asked as to whether the Ginzburg-Landau equations allow a solution for a vortex carrying an axial current.[21] It will be shown that this is indeed the case.

The dimensionless form of the Ginzburg-Landau equations is[22]

$$\begin{aligned} -\nabla\times(\nabla\times\boldsymbol{A}) &= |\Psi|^2\boldsymbol{A} + \frac{1}{2}i\kappa^{-1}\left\{\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right\} \\ \left(i\kappa^{-1}\nabla + \boldsymbol{A}\right)^2\Psi &= \Psi\left(1 - |\Psi|^2\right), \end{aligned} \tag{6.13}$$

where Ψ is the order parameter, which can be thought of as an averaged wave function of the superconducting electrons.[23]
By letting the order parameter be $\Psi = fe^{i\varphi}$, the first equation reduces to

$$-\nabla\times(\nabla\times\boldsymbol{A}) = f^2\left(\boldsymbol{A} - \kappa^{-1}\nabla\varphi\right). \tag{6.14}$$

Abrikosov introduced a scalar quantity Q which is the magnitude of the vector $\boldsymbol{A} - \kappa^{-1}\nabla\varphi$. Consider instead, the introduction of a *vector* quantity $\boldsymbol{Q} = \boldsymbol{A} - \kappa^{-1}\nabla\varphi$. Since $\nabla\times\boldsymbol{Q} = \nabla\times\boldsymbol{A} = \boldsymbol{B}$, Eq. (6.14) can be written as

$$-\nabla\times(\nabla\times\boldsymbol{Q}) = f^2\boldsymbol{Q}. \tag{6.15}$$

In cylindrical coordinates, Eq. (6.15) results in two equations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(rQ_\phi)\right\} &= f^2Q_\phi \\ \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\frac{\mathrm{d}Q_z}{\mathrm{d}r}\right\} &= f^2Q_z, \end{aligned} \tag{6.16}$$

which are valid when the flux vortices are sufficiently separated to have axial symmetry.

Note that $\boldsymbol{Q} = \boldsymbol{A} - \kappa^{-1}\nabla\varphi$ is not a true gauge transformation since φ is a phase.[24] Consider the line integral of $\boldsymbol{Q}$ around a closed path Γ,

$$\oint_\Gamma \boldsymbol{Q}\cdot\mathbf{d}\boldsymbol{l} = \oint_\Gamma \boldsymbol{A}\cdot\mathbf{d}\boldsymbol{l} - \kappa^{-1}\oint_\Gamma \nabla\varphi\cdot\mathbf{d}\boldsymbol{l}. \tag{6.17}$$

Since φ is a multi-valued function, it can be made single valued by introducing a cut defined by, say, $\phi = 0$. Now for any two points p_1 and p_2, the difference in the value of the scalar function φ is $\varphi_{p2}-\varphi_{p1} = \int_{p_1}^{p_2} \nabla\varphi\cdot\mathbf{d}\boldsymbol{l}$. If the path of integration is closed, but does not cross the cut needed to make φ single valued, the integral will vanish by Stokes' theorem. Now as $r \to \infty$, $\boldsymbol{Q} \to 0$, so that if the contour Γ is chosen at infinity, and the path of integration Γ crosses the cut, $\oint_\Gamma \nabla\varphi\cdot\mathbf{d}\boldsymbol{l} = \kappa\oint_\Gamma \boldsymbol{A}\cdot\mathbf{d}\boldsymbol{l}$. The last integral is the total flux through Γ. Since φ is a phase, it must change by 2π each time Γ is traversed. Thus, the integral $\oint_\Gamma \boldsymbol{A}\cdot\mathbf{d}\boldsymbol{l}$ must be a multiple n of a fixed quantity of flux $\Phi_0 = \frac{2\pi}{\kappa}$, so that $\oint_\Gamma \nabla\varphi\cdot\mathbf{d}\boldsymbol{l} = n\kappa\Phi_0$. It can be shown[25] that the value of this integral, while it depends on n, is not dependent on the particular path Γ. Thus, for an arbitrary path Γ, Eq.(6.17) can be written as $\oint_\Gamma \boldsymbol{Q}\cdot\mathbf{d}\boldsymbol{l} = \Phi - n\Phi_0$, where Φ is the is magnetic flux through the surface bounded by the contour Γ. n is the winding number which can be defined, for example, when the path Γ is contained in the plane $z = 0$ as

$$n = \frac{1}{2\pi}\int_0^{2\pi} \frac{\mathrm{d}\varphi(\phi)}{\mathrm{d}\phi}\,\mathrm{d}\phi.$$

The winding number n is also known as the topological charge.

Following Abrikosov, if φ is set equal to the azimuthal coordinate ϕ, the second of Eqs. (6.13) becomes

$$-\kappa^{-2}\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) + Q^2 f = f\left(1-f^2\right), \tag{6.18}$$

where, despite the formal similarity, this differs from the usual result since here $Q^2 = Q_\phi^2 + Q_z^2$. In what follows, attention will be restricted to values of r sufficiently far from the vortex core ($r > \kappa^{-1}$ in Ginzburg-Landau units) so that $f \sim 1$, and Eq. (6.18) will play no further role.

With $f = 1$ and the boundary condition that $\boldsymbol{Q} \to 0$ as $r \to \infty$, Eqs.(6.16) have the solutions

$$\begin{aligned} Q_\phi &= K_1(r) \\ Q_z &= \mathfrak{C}\, K_0(r), \end{aligned} \tag{6.19}$$

where the K_n are modified Bessel functions of the second kind, $\mathfrak{C}$ is a constant, and the constant associated with Q_ϕ has been set equal to unity so that the solution will reduce to Abrikosov's for $\mathfrak{C} = 0$. The magnetic field and current associated with $\boldsymbol{Q} = \{0, K_1(r), \mathfrak{C}\, K_0(r)\}$ are readily calculated to be[26]

$$\begin{aligned} \boldsymbol{B} &= \{0, \mathfrak{C}K_1(r), -K_0(r)\} \\ \boldsymbol{J} &= \{0, -K_1(r), -\mathfrak{C}\, K_0(r)\}. \end{aligned} \tag{6.20}$$

It is easily verified that Eqs. (6.20) *do not* satisfy, for real values of α, the force-free relation.

Note that the cosine of the angle between $\boldsymbol{J}$ and $\boldsymbol{B}$ given by Eqs. (6.20) is

$$\frac{\mathfrak{C}\left\{[K_0(r)]^2 - [K_1(r)]^2\right\}}{\left\{[K_1(r)]^2 + \mathfrak{C}^2[K_0(r)]^2\right\}^{1/2}\left\{[K_0(r)]^2 + \mathfrak{C}^2[K_1(r)]^2\right\}^{1/2}},$$

so that using the relation[27]

$$\lim_{r\to 0} K_n(r) \cong \frac{1}{2}\Gamma(n)\left(\frac{r}{2}\right)^{-n}$$

it can be seen that $\boldsymbol{J}$ and $\boldsymbol{B}$ are perpendicular for values of r large compared to unity and become parallel or anti-parallel (depending on the sign of $\mathfrak{C}$) for small values of r. The solution is not valid, of course, for values of $r \le \xi$. The variation in currents and field over this region, where Eqs. (6.16) and Eq. (6.18) must apply with $f = f(r)$, must be such as to match the axial field at $r = 0$ and the solution given by Eqs. (6.20) for $r > \xi$.

The fact that the solution given by Eqs. (6.20) does not satisfy the force-free relation raises the question of whether *any* axially symmetric force-free solution exists that decays exponentially with increasing distance into a superconductor. As will now be shown, the answer is no.

For axially symmetric fields such that $\boldsymbol{B} = \{0, B_\phi(r), B_z(r)\}$, the force-free condition can be put in the form

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(B_\phi^2 + B_z^2\right) + 2\frac{B_\phi^2}{r} = 0, \tag{6.21}$$

with α being given by,

$$\alpha(r) = \frac{1}{B_z}\frac{1}{r}\frac{\mathrm{d}(rB_\phi)}{\mathrm{d}r}. \tag{6.22}$$

As discussed in Section 3.2.3, any function $g(r) = B_\phi^2 + B_z^2$ such that $g(r) \geq 0$, $g'(r) \leq 0$, and $\frac{\mathrm{d}[r^2 g(r)]}{\mathrm{d}r} \geq 0$ will give a non-singular solution of Eq. (6.21). For distances r greater than the coherence length, the London equations are expected to govern the fields and currents and the field would have to decay exponentially if a force-free model of a vortex was possible to construct.

If one attempts to use the relations given by Eqs. (6.21) and (6.22) to find a force-free configuration for a flux vortex, with the condition that the magnitude of the field decay exponentially with the distance r into the superconductor, the last inequality given above is violated for some value of r. For example, let $g(r) = B_0 e^{-r/\lambda}$. Eq. (6.21) is easily solved to give

$$\begin{aligned} B_\phi &= B_0 e^{-r/\lambda}\left(\frac{r}{\lambda}\right)^{1/2} \\ B_z &= B_0 e^{-r/\lambda}\left(1 - \frac{r}{\lambda}\right)^{1/2}. \end{aligned} \tag{6.23}$$

B_z is imaginary for $r > \lambda$, and $\alpha(r)$ is singular at $r = \lambda$. For $r > \lambda$, $\frac{\mathrm{d}[r^2 g(r)]}{\mathrm{d}r} = 1 - \frac{r}{\lambda} < 0$. Thus a non-singular force-free model for a vortex, where the magnitude of the field decays exponentially for distances greater than the coherence length, is not possible.

It is an interesting fact that the Ginzburg-Landau equations are equivalent to ideal-fluid hydrodynamics and London electrodynamics. This was originally shown by Fröhlich.[28] For a vortex carrying a non-zero longitudinal current, it can be shown that the Ginzburg-Landau equations are equivalent to the radial pressure-balance equilibrium relation in ideal magneto-hydrodynamics. This equivalency can be used to qualitatively address the question of vortex stability.[29] To begin, rewrite Eqs. (6.16) with $f = 1$ as:

$$Q_\phi = \frac{\mathrm{d}B_z}{\mathrm{d}r}, \qquad Q_z = -\frac{1}{r}\frac{\mathrm{d}[rB_\phi]}{\mathrm{d}r}. \tag{6.24}$$

Multiplying the first of these equations by H_z, the second by H_ϕ, and subtracting the second from the first results in

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{B_\phi^2 + B_z^2}{2}\right) + \frac{B_\phi^2}{r} = B_z Q_\phi - B_\phi Q_z. \tag{6.25}$$

The right hand side of this equation is just the $\hat{r}$ component of $\boldsymbol{Q}\times\boldsymbol{B}$, the only component that does not vanish. From the vector relation $(\nabla\times\boldsymbol{H})\times\boldsymbol{H} = (\boldsymbol{H}\cdot\nabla)\boldsymbol{H} - \nabla(H^2/2)$, it is readily seen that $\boldsymbol{Q}\times\boldsymbol{B} = -\boldsymbol{J}\times\boldsymbol{B}$. If $\boldsymbol{B}\times\boldsymbol{Q}$ is viewed as the gradient of a pressure P, Eq. (6.25) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(P + \frac{B_\phi^2 + B_z^2}{2}\right) + \frac{B_\phi^2}{r} = 0. \tag{6.26}$$

For the solutions for $\boldsymbol{Q}$ and $\boldsymbol{B}$ given by Eqs. (6.19) and (6.20) the $\hat{r}$-component of $\boldsymbol{Q}\times\boldsymbol{B}$ may be calculated to be

$$\hat{r}\cdot\left(\boldsymbol{Q}\times\boldsymbol{B}\right) = -\left(1 + \mathfrak{C}^2\right)K_1(r)K_0(r) = \frac{\left(1 + \mathfrak{C}^2\right)}{2}\frac{\mathrm{d}B_z^2}{\mathrm{d}r}. \tag{6.27}$$

P is then

$$P = -\frac{\left(1 + \mathfrak{C}^2\right)}{2}B_z^2. \tag{6.28}$$

Note that ∇P is in the positive $\hat{r}$-direction. Eq. (6.26) is the radial pressure-balance equilibrium relation in ideal magneto-hydrodynamics.[30] For $\mathfrak{C} = 0$, the latter is satisfied identically.

There are some general constraints implied by the form given by Eq. (6.25) for Eqs. (6.16). If one defines

$$g(r) = P + \frac{B_\phi^2 + B_z^2}{2}, \tag{6.29}$$

Eq. (6.26) can be written as

$$B_\phi^2 = -r\frac{\mathrm{d}}{\mathrm{d}r}g(r). \tag{6.30}$$

From Eqs. (6.29) and (6.30),

$$B_z^2 = 2[g(r) - P] + r\frac{\mathrm{d}}{\mathrm{d}r}g(r). \tag{6.31}$$

Thus, B_ϕ and B_z will be real if $\frac{\mathrm{d}}{\mathrm{d}r}g(r) \leq 0$ and if $\frac{\mathrm{d}[r^2 g(r)]}{\mathrm{d}r} \geq 2rP$. While the solution given by Eqs. (6.20) and Eq. (6.28) satisfy both these inequalities as well as $g(r) \geq 0$, it is also true that $\frac{\mathrm{d}[r^2 g(r)]}{\mathrm{d}r} \leq 0$. However, unlike the case of a force-free field, the solution given by Eqs. (6.20) is regular for all values of r.

The equivalency of the Ginzburg-Landau equations for a vortex carrying a non-zero longitudinal current and the radial pressure-balance equilibrium relation in ideal magneto-hydrodynamics means that the full panoply of techniques developed in the latter field are available to address the question of vortex stability. And, while this is a difficult problem in general, it is clear that a vortex will be subject to a kink instability for very small amounts

of twisting. The vortex can be expected to become unstable[31] at a value of axial current such that $\frac{\langle B_\phi \rangle}{\langle B_z \rangle} \approx 2\pi\lambda/L$, where λ is the radius of a vortex (approximately the penetration depth) and L the length of the vortex. Since the vortex is subject to a kink instability for very small values of $\mathfrak{C}$, the azimuthal flux will be small compared to the quantized axial flux. Note also that the azimuthal flux is not quantized.[32]

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Appendix 1

FORCE-FREE FIELDS AND ELECTROMAGNETIC WAVES

That electromagnetic waves could exist with $\boldsymbol{E}$ parallel to $\boldsymbol{B}$ is counter to physical intuition and has caused some past controversy in the literature. Such waves correspond to standing waves and have a vanishing Poynting vector. They, and their relation to the force-free condition, are treated in the first section of this Appendix.

Perhaps somewhat surprisingly, the force-free relation also makes an appearance when considering the propagation of electromagnetic waves in optically active media. Optical activity in such media is a result of the chiral nature of the molecules constituting the medium. They have a left or right handedness. As a result, left and right circularly polarized light have different phase velocities in such materials. Chiral media exhibit the phenomena of circular dichroism, where there is a differential absorption of left and right circularly polarized plane waves, and optical rotary dispersion, where the polarization of a transmitted plane wave is rotated with respect to the polarization of the incident wave.

It is an interesting fact that only plane circularly polarized waves can propagate in optically active media without a change in polarization. Optically active media can be characterized by constitutive relations such that $\boldsymbol{D}$ depends on the curl of $\boldsymbol{E}$ as well as $\boldsymbol{E}$, and $\boldsymbol{B}$ depends on the curl of $\boldsymbol{H}$ as well as $\boldsymbol{H}$. The second section of this Appendix gives an introduction to this subject.

A1.1 Electromagnetic Fields with $\boldsymbol{E}$ Parallel to $\boldsymbol{B}$

Chu and Ohkawa,[1] called attention to the existence of a general class of transverse electromagnetic waves with $\boldsymbol{E} \parallel \boldsymbol{B}$, and gave as an example the following solution to the free-space Maxwell equations:

$$\begin{aligned} \boldsymbol{k} &= k\,(0, 0, 1), \\ \boldsymbol{A} &= A\,(\sin kz, \cos kz, 0)\cos \omega t, \\ \boldsymbol{E} &= (\omega A/c)\,(\sin kz, \cos kz, 0)\sin \omega t, \\ \boldsymbol{B} &= kA\,(\sin kz, \cos kz, 0)\cos \omega t. \end{aligned} \tag{A1.1}$$

This solution corresponds to a standing wave which can be represented as two circular polarized waves with opposite directions of propagation. Note that $\boldsymbol{A}$ is also parallel to $\boldsymbol{E}$ and $\boldsymbol{B}$, and that since $\boldsymbol{E} \parallel \boldsymbol{B}$, the Poynting vector $\boldsymbol{E}\times\boldsymbol{B}$ vanishes.

As pointed out by Brownstein,[2] the field given in Eq. (A1.1) is simply a superposition

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{E}_1 + \boldsymbol{E}_2 + \boldsymbol{E}_3 + \boldsymbol{E}_4) \tag{A1.2}$$

and

$$\boldsymbol{B} = \tfrac{1}{2}(\boldsymbol{B}_1 + \boldsymbol{B}_2 + \boldsymbol{B}_3 + \boldsymbol{B}_4) \tag{A1.3}$$

of the plane polarized traveling waves:

$$\begin{aligned} \boldsymbol{E}_1 &= kA\cos(kz - \omega t)\,\hat{\boldsymbol{i}}; & \boldsymbol{B}_1 &= kA\cos(kz - \omega t)\,\hat{\boldsymbol{j}}; \\ \boldsymbol{E}_2 &= -kA\sin(kz - \omega t)\,\hat{\boldsymbol{j}}; & \boldsymbol{B}_2 &= kA\sin(kz - \omega t)\,\hat{\boldsymbol{i}}; \\ \boldsymbol{E}_3 &= -kA\cos(kz + \omega t)\,\hat{\boldsymbol{i}}; & \boldsymbol{B}_3 &= kA\cos(kz + \omega t)\,\hat{\boldsymbol{j}}; \\ \boldsymbol{E}_4 &= kA\sin(kz + \omega t)\,\hat{\boldsymbol{j}}; & \boldsymbol{B}_4 &= kA\sin(kz + \omega t)\,\hat{\boldsymbol{i}}. \end{aligned} \tag{A1.4}$$

Fields having $\boldsymbol{E} \parallel \boldsymbol{B}$ are closely related to the constant α force-free field equations, and it will be shown here that solutions to the force-free equations can be used to generate such fields. Maxwell's equations in free space (in natural units where $\varepsilon_0 = \mu_0 = 1$ and $c = 1$) are:

$$\begin{aligned} \dot{\boldsymbol{E}} &= \nabla\times\boldsymbol{B}, \\ \dot{\boldsymbol{B}} &= -\nabla\times\boldsymbol{E}, \\ \nabla\cdot\boldsymbol{B} &= \nabla\cdot\boldsymbol{E} = 0. \end{aligned} \tag{A1.5}$$

The condition that $\boldsymbol{E} \parallel \boldsymbol{B}$ means that $\boldsymbol{E} = \beta\boldsymbol{B}$. If β is assumed to be a constant, it is readily seen that Maxwell's equations determine β as $\beta = \pm i$, so there are no real solutions.

If β is assumed to be only a function of time (the only case that will be considered here), differentiation the relationship $\boldsymbol{E} = \beta\boldsymbol{B}$ with respect to time, and combination with Eqs. (A1.5) immediately yields the relation

$$\dot{\boldsymbol{E}} = \nabla\times\boldsymbol{B} = \dot{\beta}\boldsymbol{B} + \beta\dot{\boldsymbol{B}} = \dot{\beta}\boldsymbol{B} - \beta\nabla\times\boldsymbol{E} = \dot{\beta}\boldsymbol{B} - \beta^2\nabla\times\boldsymbol{B}, \tag{A1.6}$$

so that

$$\nabla\times\boldsymbol{B} = \frac{\dot{\beta}}{\beta^2 + 1}\boldsymbol{B}. \tag{A1.7}$$

Taking the curl of $\boldsymbol{E} = \beta\boldsymbol{B}$ and using Maxwell's equations gives

$$\nabla\times\boldsymbol{E} = -\dot{\boldsymbol{B}} = \beta\nabla\times\boldsymbol{B} = \beta\dot{\boldsymbol{E}} = \beta(\dot{\beta}\boldsymbol{B} + \beta\dot{\boldsymbol{B}}), \tag{A1.8}$$

which results in the partial differential equation for $\boldsymbol{B}$

$$(\beta^2 + 1)\dot{\boldsymbol{B}} + \beta\dot{\beta}\boldsymbol{B} = 0. \tag{A1.9}$$

Equation (A1.7) means that if $\boldsymbol{B}$ satisfies $\boldsymbol{E} = \beta(t)\boldsymbol{B}$, then $\boldsymbol{B}$ must also satisfy the force-free condition.

Perhaps somewhat surprisingly, the function $\dot{\beta}(\beta^2 + 1)^{-1}$ in Eq. (A1.7) is a constant. This can be seen as follows: If Eq. (A1.9) is separated by assuming a solution of the form $\boldsymbol{B}(\boldsymbol{r}, t) = \boldsymbol{B}(\boldsymbol{r})f(t)$, where $\boldsymbol{B}(\boldsymbol{r})$ is time independent, one obtains the ordinary differential equation for f

$$\frac{\dot{f}}{f} + \frac{\beta\dot{\beta}}{(\beta^2 + 1)} = 0. \tag{A1.10}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln(\beta^2 + 1) = \frac{2\beta\dot{\beta}}{(\beta^2 + 1)}, \tag{A1.11}$$

Eq. (A1.10) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\ln f = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ln(\beta^2 + 1). \tag{A1.12}$$

This is readily integrated to give

$$f(t) = (\beta^2 + 1)^{-1/2}, \tag{A1.13}$$

where an arbitrary constant of integration has been ignored. This allows $\boldsymbol{B}(\boldsymbol{r}, t)$ to be written as

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{\boldsymbol{B}(\boldsymbol{r})}{(\beta^2 + 1)^{1/2}}, \tag{A1.14}$$

which, using Eq. (A1.7), gives

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \frac{\dot{\beta}}{\beta^2 + 1} \boldsymbol{B}(\boldsymbol{r}). \tag{A1.15}$$

Since $\boldsymbol{B}(\boldsymbol{r})$ is time independent, the expression $\dot{\beta}(\beta^2 + 1)^{-1}$ must be equal to a constant α. Thus, any magnetostatic solution to the force-free field equations can be used to construct a solution to Maxwell's equations with $\boldsymbol{E}$ parallel to $\boldsymbol{B}$. The techniques discussed in Chapter 3 for generating solutions to the constant α force-free field equations from the wave equation are therefore applicable.

The constancy of $\dot{\beta}(\beta^2 + 1)^{-1}$ results in the differential equation

$$\dot{\beta} - \alpha\beta^2 - \alpha = 0, \tag{A1.16}$$

which is a form of the Riccati equation. Substituting $\beta = -\dot{u}/\alpha u$ results in $u = Ae^{i\alpha t} + Be^{-i\alpha t}$, so that

$$\beta = i\frac{Be^{-i\alpha t} - Ae^{i\alpha t}}{Be^{-i\alpha t} + Ae^{i\alpha t}}. \tag{A1.17}$$

The case of vanishing A or B gives $\beta = \pm i$, the same as the previous case where β was constant. If $A = \pm B$, then $\beta = \tan \alpha t$ or $\beta = \cot \alpha t$ respectively.

It is interesting to compute the helicity and energy densities from Eqs. (A1.1). These may be obtained by inspection, and are

$$\boldsymbol{A} \cdot \boldsymbol{B} = kA^2 \cos^2 \omega t \tag{A1.18}$$

and

$$B^2 = k^2 A^2 \cos^2 \omega t. \tag{A1.19}$$

If these are averaged over time, one obtains the following relationship between the average energy and helicity densities:

$$\langle \varepsilon \rangle = \frac{k}{2} \langle \boldsymbol{A} \cdot \boldsymbol{B} \rangle. \tag{A1.20}$$

The energy density of the field is in general given by $\varepsilon = \frac{1}{2} B^2$. Equations (A1.1) imply that $\boldsymbol{B} = k\boldsymbol{A}$, so that the energy density for this type of field can be written as $\varepsilon = \frac{k}{2} \boldsymbol{B} \cdot \boldsymbol{A}$, and Eq. (A1.20) results from taking the time average. If, in addition, k is identified with α, and Eq. (A1.7) is used to substitute for $\boldsymbol{B}$, the energy density becomes

$$\varepsilon = \frac{1}{2} (\nabla \times \boldsymbol{B}) \cdot \boldsymbol{A}. \tag{A1.21}$$

The first of Eqs. (A1.5) then allow this to be written as

$$\varepsilon = \frac{1}{2} \dot{\boldsymbol{E}} \cdot \boldsymbol{A}. \tag{A1.22}$$

Now $\dot{\boldsymbol{E}}$ is the displacement current in the natural units defined just after Eqs. (A1.4). In the magnetostatic case, the energy density can be written as half the inner product of the current and the vector potential. Equation (A1.22) shows that the same is true for the specialized fields of this Appendix, with the displacement current playing the role of the real current in magnetostatics.

The time average value of the energy density can also be written in terms of the time independent fields $\boldsymbol{B}(\boldsymbol{r})$ and $\boldsymbol{A}(\boldsymbol{r})$. Substituting $\boldsymbol{D} = \beta \boldsymbol{B}$ and Eq. (A1.14) for $\boldsymbol{B}$, the energy density becomes

$$\varepsilon = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left[\frac{\beta}{(\beta^2 + 1)^{1/2}} \right] \boldsymbol{B}(\boldsymbol{r}) \cdot \boldsymbol{A}. \tag{A1.23}$$

The vector potential can be obtained from Eq. (A1.14) by writing the latter equation as

$$\nabla \times \boldsymbol{A} = \frac{\nabla \times \boldsymbol{A}(\boldsymbol{r})}{(\beta^2 + 1)^{1/2}}, \tag{A1.24}$$

or

$$\nabla \times \left[\boldsymbol{A} - \frac{\boldsymbol{A}(\boldsymbol{r})}{(\beta^2 + 1)^{1/2}} \right] = 0. \tag{A1.25}$$

This means that, except for an arbitrary gradient, $\boldsymbol{A}$ may be written as

$$A = \frac{A(r)}{(\beta^2+1)^{1/2}}. \tag{A1.26}$$

Since all calculations here are local, the arbitrary gradient may be ignored. But care must be taken when computing the total energy because, if the space is not simply connected and the field such that the normal component of $\boldsymbol{B}$ vanishes on the boundary, the gradient may have a physical significance, as discussed in Chapters 4 and 5.

The time derivative in Eq. (A1.23) is

$$\frac{d}{dt}\left[\frac{\beta}{(\beta^2+1)^{1/2}}\right] = \frac{\dot{\beta}}{(\beta^2+1)^{3/2}}. \tag{A1.27}$$

Using this result, and Eq. (A1.26), in Eq. (A1.23) allows the energy density to be written as

$$\varepsilon = \frac{1}{2}\frac{\dot{\beta}}{(\beta^2+1)^2}\boldsymbol{B}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}) = \frac{1}{2}\frac{\alpha}{(\beta^2+1)}\boldsymbol{B}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}). \tag{A1.28}$$

For the case derived earlier, where $\beta = \tan\alpha t$ or $\beta = \cot\alpha t$, the energy density becomes

$$\varepsilon = \alpha\frac{1}{2}\left\{\begin{matrix}\cos^2\alpha t\\ \sin^2\alpha t\end{matrix}\right\}\boldsymbol{B}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}). \tag{A1.29}$$

If Eq. (A1.29) is averaged over time, the result is

$$\langle\varepsilon\rangle = \frac{\alpha}{4}\boldsymbol{B}(\boldsymbol{r})\cdot\boldsymbol{A}(\boldsymbol{r}). \tag{A1.30}$$

A1.2 The Propagation of Electromagnetic Waves in Optically Active Media

An exact solution to the problem of light scattering by an optically active sphere was first given by Bohren.[3] He used the constitutive relations

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} + \alpha\varepsilon\nabla\times\boldsymbol{E} \quad \text{and} \quad \boldsymbol{B} = \mu\boldsymbol{H} + \beta\mu\nabla\times\boldsymbol{H}, \tag{A1.31}$$

where ε, μ, α, and β are scalar phenomenological coefficients, to characterize the medium. When $\alpha = \beta$, β is known as the chirality parameter, and since $\boldsymbol{D}$ and $\boldsymbol{E}$ are polar vectors, β must be a pseudoscalar.

It is interesting to note that Post,[4,5] by requiring all equations to be generally covariant, was able to derive the set of constitutive relations

$$\boldsymbol{D} = \varepsilon\boldsymbol{E} + i\xi\boldsymbol{B}, \qquad \boldsymbol{B} = \mu_P(\boldsymbol{H} - i\xi\boldsymbol{E}). \tag{A1.32}$$

This is all the more remarkable when α is set equal to β and a harmonic time dependence is assumed for the fields in Eqs. (A1.31). When the result compared to Eqs. (A1.32), the two sets of constitutive relations will be the same if $\mu_P = \mu/(1-\omega^2\beta^2\varepsilon\mu)$.

To solve the scattering problem, Bohren introduced a transformation of the fields such that the resulting combinations of $\boldsymbol{E}$ and $\boldsymbol{H}$ satisfy the force-free relation. Asghar and Lakhtakia[6] have termed the fields that satisfy this relation Beltrami fields. To see how the transformation arises, assume a time dependence of $e^{-i\omega t}$. Taking the time derivative of Eqs. (A1.31) and using the Maxwell relations

$$\nabla\times\boldsymbol{H} = \frac{\partial\boldsymbol{D}}{\partial t}, \qquad \nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t} \tag{A1.33}$$

results in the two relations

$$\begin{aligned}\nabla\times\boldsymbol{H} &= \frac{-i\omega}{1-\omega^2\alpha\beta\varepsilon\mu}(\varepsilon\boldsymbol{E} + i\omega\alpha\varepsilon\mu\boldsymbol{H}),\\ \nabla\times\boldsymbol{E} &= \frac{i\omega}{1-\omega^2\alpha\beta\varepsilon\mu}(\mu\boldsymbol{H} - i\omega\beta\varepsilon\mu\boldsymbol{E}).\end{aligned} \tag{A1.34}$$

These may be written as

$$\nabla\times\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = K\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}, \tag{A1.35}$$

where K is the matrix

$$K = \frac{i\omega}{1-\omega^2\alpha\beta\varepsilon\mu}\begin{pmatrix}-i\omega\beta\varepsilon\mu & \mu\\ -\varepsilon & -i\omega\alpha\varepsilon\mu\end{pmatrix}. \tag{A1.36}$$

The remaining Maxwell equations give

$$\nabla\cdot\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix} = 0. \tag{A1.37}$$

Consider the linear transformation of the electromagnetic field by a matrix A

$$\begin{pmatrix}Q_L\\ Q_R\end{pmatrix} = A^{-1}\begin{pmatrix}\boldsymbol{E}\\ \boldsymbol{H}\end{pmatrix}. \tag{A1.38}$$

Under such a transformation Eq. (A1.35) becomes

$$\nabla\times\begin{pmatrix}\boldsymbol{Q}_L\\ \boldsymbol{Q}_R\end{pmatrix} = (A^{-1}KA)\begin{pmatrix}\boldsymbol{Q}_L\\ \boldsymbol{Q}_R\end{pmatrix}. \tag{A1.39}$$

Now let the matrix A be defined as

$$A = \begin{pmatrix}1 & a_R\\ a_L & 1\end{pmatrix}, \tag{A1.40}$$

where

$$\begin{aligned} a_R &= -i\left\{\frac{k_R(1-\alpha\beta\omega^2\varepsilon\mu)+\alpha\omega^2\varepsilon\mu}{\omega\varepsilon}\right\}\\ a_L &= -i\left\{\frac{k_L(1-\alpha\beta\omega^2\varepsilon\mu)-\beta\omega^2\varepsilon\mu}{\omega\mu}\right\}\end{aligned} \tag{A1.41}$$

and

$$\begin{aligned} k_R &= \omega(\varepsilon\mu)^{1/2}\left\{\frac{\left[1+(\alpha-\beta)^2\omega^2\varepsilon\mu/4\right]^{1/2}-(\alpha+\beta)\omega(\varepsilon\mu)^{1/2}/2}{1-\alpha\beta\omega^2\varepsilon\mu}\right\}\\ k_L &= \omega(\varepsilon\mu)^{1/2}\left\{\frac{\left[1+(\alpha-\beta)^2\omega^2\varepsilon\mu/4\right]^{1/2}+(\alpha+\beta)\omega(\varepsilon\mu)^{1/2}/2}{1-\alpha\beta\omega^2\varepsilon\mu}\right\}.\end{aligned} \tag{A1.42}$$

Under the transformation $A^{-1}KA$, K will be diagonal and Eq. (A1.39) will have the form

$$\nabla\times\begin{pmatrix}\boldsymbol{Q}_L\\ \boldsymbol{Q}_R\end{pmatrix} = \begin{pmatrix}k_L & 0\\ 0 & -k_R\end{pmatrix}\begin{pmatrix}\boldsymbol{Q}_L\\ \boldsymbol{Q}_R\end{pmatrix}. \tag{A1.43}$$

A^{-1} may readily be computed to be

$$A^{-1} = \frac{1}{1-a_La_R}\begin{pmatrix}1 & -a_R\\ -a_L & 1\end{pmatrix}, \tag{A1.44}$$

so that $\boldsymbol{Q}_L$ and $\boldsymbol{Q}_R$ are given by

$$\begin{aligned} \boldsymbol{Q}_L &= \frac{1}{1-a_La_R}(\boldsymbol{E}-a_R\boldsymbol{H})\\ \boldsymbol{Q}_R &= \frac{1}{1-a_La_R}(\boldsymbol{H}-a_L\boldsymbol{E}).\end{aligned} \tag{A1.45}$$

Equation (A1.43) may then be written in the compact form

$$\nabla\times\boldsymbol{Q} = k\boldsymbol{Q}, \tag{A1.46}$$

where $k = k_L$ for $\boldsymbol{Q} = \boldsymbol{Q}_L$ and $k = -k_R$ for $\boldsymbol{Q} = \boldsymbol{Q}_R$. Equation (A1.46) is, of course, the force-free relation, and since k is constant, $\boldsymbol{Q}$ will also satisfy the vector wave equation

$$\nabla^2\boldsymbol{Q} + k^2\boldsymbol{Q} = 0. \tag{A1.47}$$

In addition, $\boldsymbol{Q}$ also satisfies the divergence condition $\nabla\cdot\boldsymbol{Q} = 0$.

Plane wave solutions to Eq. (A1.46) are either left or right circularly polarized waves. This can be seen by letting $\boldsymbol{Q} = \boldsymbol{Q}_0e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, where $\boldsymbol{k} = k\hat{\boldsymbol{e}}$, $\boldsymbol{Q}_0$ is a constant vector, and k is the same as in Eq. (A1.46). This is a wave propagating in the $\hat{\boldsymbol{e}}$ direction. Substitution into Eq. (A1.46) results in

$$\nabla\times\boldsymbol{Q} = \nabla\times\boldsymbol{Q}_0e^{ik\hat{\boldsymbol{e}}\cdot\boldsymbol{x}} = \nabla e^{ik\hat{\boldsymbol{e}}\cdot\boldsymbol{x}}\times\boldsymbol{Q}_0 + e^{ik\hat{\boldsymbol{e}}\cdot\boldsymbol{x}}\nabla\times\boldsymbol{Q}_0 = k\boldsymbol{Q}. \tag{A1.48}$$

The second term on the right hand side of the first equal sign vanishes since $\boldsymbol{Q}_0$ is a constant vector, and taking the gradient of the exponential term results in

$$ike^{ik\hat{\boldsymbol{e}}\cdot\boldsymbol{x}}\,\hat{\boldsymbol{e}}\times\boldsymbol{Q}_0 = ik\hat{\boldsymbol{e}}\times\boldsymbol{Q} = k\boldsymbol{Q}. \tag{A1.49}$$

If $\boldsymbol{Q}$ is taken to be $\boldsymbol{Q}_R$, then $k = -k_R$ on the right hand side of Eq. (A1.49), but if the wave is to be propagating in the $\hat{\boldsymbol{e}}$ direction, k must be $+k_R$ in the exponential. Equation (A1.49) may therefore be written as

$$i\,\hat{\boldsymbol{e}}\times\boldsymbol{Q} = \pm\boldsymbol{Q}, \tag{A1.50}$$

where the plus sign corresponds to $\boldsymbol{Q} = \boldsymbol{Q}_L$ and the minus sign to $\boldsymbol{Q} = \boldsymbol{Q}_R$. The requirement that $\nabla\cdot\boldsymbol{Q} = 0$ imposes the condition

$$\nabla\cdot(\boldsymbol{Q}_0e^{ik\hat{\boldsymbol{e}}\cdot\boldsymbol{x}}) = \boldsymbol{Q}_0\cdot\nabla e^{ik\hat{\boldsymbol{e}}\cdot\boldsymbol{x}} = ik\hat{\boldsymbol{e}}\cdot\boldsymbol{Q} = 0, \tag{A1.51}$$

or,

$$\hat{e}\cdot Q = 0. \tag{A1.52}$$

Q is therefore in a plane perpendicular to $\hat{e}$. If one chooses orthogonal axes $\hat{e}_1$ and $\hat{e}_2$ in this plane, Q may be written as

$$Q = Q_1\hat{e}_1 + Q_2\hat{e}_2. \tag{A1.53}$$

The condition given by Eq. (A1.50) can then be written out explicitly in terms of the coordinates $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}$ as

$$i\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e} \\ 0 & 0 & 1 \\ Q_1 & Q_2 & 0 \end{vmatrix} = i(\hat{e}_2Q_1 - \hat{e}_1Q_2) = \pm(\hat{e}_1Q_1 + \hat{e}_2Q_2). \tag{A1.54}$$

This means that $Q_2 = \pm iQ_1$ so that Eq. (A1.53) can be written as

$$Q = Q_1(\hat{e}_1 \pm i\hat{e}_2). \tag{A1.55}$$

This is a circularly polarized wave, $(\hat{e}_1 + i\hat{e}_2)$ corresponding to a right handed polarization and $(\hat{e}_1 - i\hat{e}_2)$ to a left handed polarization.

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Appendix 2

PROOF OF THE JACOBI POLYNOMIAL IDENTITIES

Before proving the Jacobi polynomial identities used in Chapter 3, it may be useful to provide a short précis of some relevant aspects of the Jacobi polynomials themselves. The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ satisfy the differential equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0. \tag{A2.1}$$

Rather than working directly with this equation, it is possible to apply what are known as elementary transformations, which in general can be applied to any homogeneous linear differential equation of the second order, in order to obtain a differential equation having a more convenient form. In particular, Eq. (A2.1) can be brought into the form

$$\frac{d^2u}{dx^2} + \left\{\frac{1}{4}\frac{1-\alpha^2}{(1-x)^2} + \frac{1}{4}\frac{1-\beta^2}{(1+x)^2} + \frac{n(n+\alpha+\beta+1)+(\alpha+1)(\beta+1)/2}{1-x^2}\right\}u = 0, \tag{A2.2}$$

where

$$u(x) = (1-x)^{(\alpha+1)/2}(1+x)^{(\beta+1)/2}P_n^{(\alpha,\beta)}(x). \tag{A2.3}$$

The case where $\alpha = \pm 1$ and $n = 0, 1, 2, \ldots$ is used in Chapter 3.

The scalar product of two real functions $f(x)$ and $g(x)$ is defined by the (Stieltjes-Lebesgue) integral

$$(f, g) = \int_a^b f(x)g(x)w(x)\,dx, \tag{A2.4}$$

where the weight function $w(x)$ is a non-negative Lebesgue measurable function such that $\int_a^b w(x)\,dx > 0$. The Jacobi polynomials are orthogonal on the interval $-1 \le x \le 1$ with the weight function $w(x) = (1-x)^\alpha (1+x)^\beta$. The integrability of $w(x)$ is assured by requiring that $\alpha > -1$ and $\beta > -1$.

When $\alpha = \beta$, the Jacobi polynomials are known as the Gegenbauer or ultraspherical polynomials. The relationship is given by

$$P_n^{(\lambda)}(x) = \frac{\Gamma(\lambda+1/2)}{\Gamma(2\lambda)} \frac{\Gamma(n+2\lambda)}{\Gamma(n+\lambda+1/2)} P_n^{(\lambda-1/2,\lambda-1/2)}(x). \tag{A2.5}$$

In Chapter 3, $P_n^{(\lambda)}(x)$ is designated by $C_n^\lambda(x)$ in order to be consistent with the notation used by Chandrasekhar. Another notation that is often used for the Gegenbauer polynomials is $C_n^{(\lambda)}(x)$. If $\lambda = 0$, one has the Chebyshev polynomials $C_n^{(0)}(x) = (2/n)T_n(x)$.

Another case of interest is when $\alpha = \beta = 0$. These are the Legendre polynomials $P_n(x) = P_n^{(0,0)}(x)$. The associated Legendre polynomials, except for an arbitrary constant, are

$$P_n^m(x) = \left(1-x^2\right)^{m/2} P_{n-m}^{(m+1/2)}(x). \tag{A2.6}$$

The generalized Jacobi polynomials used in Chapter 3 are obtained by extending the Jacobi polynomials to arbitrary complex values of the parameters α and β. It can then be shown,[1] that if $n \ge 0$ is an integer, there are only two cases for which the general solution to Eq. (A2.1) is a polynomial. These are characterized by the following sets of conditions:

(1) α, β are negative integers, and $\alpha \ge -n$, $\beta \ge -n$, $\alpha + \beta \le -n - 1$, $n \ge 1$;
(2) α, β are negative integers, and $\alpha < -n$, $\beta < -n$, $n \ge 0$.

The identities used in Chapter 3 can now be proven. In order to prove Eq. (3.68), consider the general relation

$$(1-\mu)P_n^{(\alpha+1,\beta)}(\mu) + (1+\mu)P_n^{(\alpha,\beta+1)}(\mu) = 2P_n^{(\alpha,\beta)}(\mu). \tag{A2.7}$$

Setting $\alpha = 0$, $\beta = -1$ and $\alpha = -1$, $\beta = 0$ results in two equations which, when added, yield

$$(1-\mu)P_n^{(1,-1)}(\mu) + (1+\mu)P_n^{(-1,1)}(\mu) + 2P_n^{(0,0)}(\mu) - 2\left[P_n^{(0,-1)}(\mu) + P_n^{(-1,0)}(\mu)\right] = 0, \tag{A2.8}$$

where $P_n^{(0,0)}(\mu)$ are the Legendre polynomials $P_n(\mu)$. To show that the last term in brackets is equivalent to $P_n(\mu)$, consider the two relations[2]

$$(\alpha+\beta+2n)P_n^{(\alpha,\beta-1)}(\mu) = (\alpha+\beta+n)P_n^{(\alpha,\beta)}(\mu) + (\alpha+n)P_{n-1}^{(\alpha,\beta)}(\mu) \tag{A2.9}$$

and

$$(\alpha+\beta+2n)P_n^{(\alpha-1,\beta)}(\mu) = (\alpha+\beta+n)P_n^{(\alpha,\beta)}(\mu) - (\beta+n)P_{n-1}^{(\alpha,\beta)}(\mu). \tag{A2.10}$$

Setting $\alpha = \beta = 0$ results in two relations which, when combined, give the desired result $P_n^{(0,-1)}(\mu) + P_n^{(-1,0)}(\mu) = P_n^{(0,0)}(\mu) = P_n(\mu)$, which proves Eq. (3.68).

In Eq. (3.69), the polynomial $P_{n+1}^{(-1,-1)}(\mu)$ satisfies the differential equation

$$D^2 P_{n+1}^{(-1,-1)}(\mu) + \frac{n(n+1)}{1-\mu^2} P_{n+1}^{(-1,-1)}(\mu) = 0, \tag{A2.11}$$

where $D^n = \dfrac{d^n}{d\mu^n}$.

Now in general

$$D^k P_n^{(\alpha,\beta)}(\mu) = 2^{-k}(1+\alpha+\beta+n)_k P_{n-k}^{(\alpha+k,\beta+k)}(\mu), \qquad 0 < k \le n \tag{A2.12}$$

where $(1+\alpha+\beta+n)_k = (1+\alpha+\beta+n)(2+\alpha+\beta+n)\ldots(1+\alpha+\beta+n+k-1)$. If, in the last equation, $\alpha = \beta = -1$, $k = 2$ and n is set equal to $n+1$, one obtains

$$D^2 P_{n+1}^{(-1,-1)}(\mu) = \frac{1}{4} n(n+1) P_{n-1}^{(1,1)}(\mu). \tag{A2.13}$$

Combining this result with the differential equation given by Eq. (A2.11) proves Eq. (3.69).

The proof of Eq. (3.70) uses the two relations

$$\frac{1}{2}(2n+\alpha+\beta+2)(1-\mu)P_n^{(\alpha+1,\beta)}(\mu) = (n+\alpha+1)P_n^{(\alpha,\beta)}(\mu) - (n+1)P_{n+1}^{(\alpha,\beta)}(\mu) \tag{A2.14}$$

and

$$\frac{1}{2}(2n+\alpha+\beta+2)(1+\mu)P_n^{(\alpha,\beta+1)}(\mu) = (n+\beta+1)P_n^{(\alpha,\beta)}(\mu) + (n+1)P_{n+1}^{(\alpha,\beta)}(\mu), \tag{A2.15}$$

with $\alpha = \beta = -1$, to obtain the two equations

$$n(1-\mu)P_n^{(0,-1)}(\mu) = nP_n^{(-1,-1)}(\mu) - (n+1)P_{n+1}^{(-1,-1)}(\mu) \tag{A2.16}$$

and

$$n(1+\mu)P_n^{(-1,0)}(\mu) = nP_n^{(-1,-1)}(\mu) + (n+1)P_{n+1}^{(-1,-1)}(\mu). \tag{A2.17}$$

Subtracting Eq. (A2.16) from Eq. (A2.17) results in

$$P_{n+1}^{(-1,-1)}(\mu) = \frac{n}{2(n+1)}\left[(1+\mu)P_n^{(-1,0)}(\mu) - (1-\mu)P_n^{(0,-1)}(\mu)\right]. \tag{A2.18}$$

Consider the bracketed term on the right hand side of Eq. (A2.18). By setting $\alpha = -1$, $\beta = 0$ and $\alpha = 0$, $\beta = -1$ in Eq. (A2.7), two equations are obtained which, when subtracted, result in

$$\left[P_n^{(-1,0)}(\mu) - P_n^{(0,-1)}(\mu)\right] = \frac{1}{2}\left[-2\mu P_n^{(0,0)}(\mu) + (1+\mu)P_n^{(-1,1)}(\mu) - (1-\mu)P_n^{(1,-1)}(\mu)\right]. \tag{A2.19}$$

The bracketed term of Eq. (A2.18) can be expanded and combined with Eq. (A2.19) to give

$$P_{n+1}^{(-1,-1)}(\mu) = \frac{n}{2(n+1)}\frac{1}{2}\left[(1+\mu)P_n^{(-1,1)}(\mu) - (1-\mu)P_n^{(1,-1)}(\mu)\right]. \tag{A2.20}$$

Eq. (3.70) is then obtained from Eq. (A2.20) by using Eq. (3.68), which was proved earlier.

Eqs. (3.71), (3.72), and (3.73) are obtained by substitution between Eqs. (3.68), (3.69), and (3.70).

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Appendix 3

SEPARATION OF THE WAVE EQUATION, CYCLIDES, AND BOUNDARY CONDITIONS

This Appendix consists of three sections. The first is concerned with the separation of the wave equation, and introduces cylindrical and rotational coordinate systems as defined by Moon and Spencer.[1] The rotational systems include toroidal coordinates within which the Helmholtz equation will not separate; fortunately, Weston's work discussed in Chapter 3 is applicable to these coordinates. Toroidal coordinates belong to the class of coordinates associated with confocal cyclides. While cyclides were widely known in the late nineteenth and early twentieth century, this is no longer the case and a discussion here may be of value. This is the subject of the second section. The third section discusses the imposition of boundary conditions on both the Helmholtz and vector wave equations. The subject of boundary conditions has an extensive literature, and it will only be possible to touch on a few aspects of the problem here. The use of Green's functions and the application of the techniques of conformal mapping to deal with geometrically complicated boundaries is well beyond the scope of this Appendix. This is also true of a comprehensive discussion of eigenfunctions and their uses. One of the most complete discussions of boundary conditions in the physics literature has been given by Morse and Feshbach.[2]

A3.1 Separation of the Wave Equation

Partial differential equations are often solved by using the method of separation of variables. This method factors the original partial differential equation, which has several independent variables, into a set of ordinary differential equations, each involving one independent variable. The factorization depends on the coordinate system used, and can only be done in a limited number of systems. Different partial differential equations are factorable in different sets of coordinate systems.

If ξ_1, ξ_2, ξ_3 represent the coordinates of a curvilinear coordinate system, a completely separated solution can be written as $\Psi = F_1(\xi_1)F_2(\xi_2)F_3(\xi_3)$. It may also be possible to find coordinate systems within which an incomplete factorization can be achieved so as to allow the solution to be written as $\Psi = F_1(\xi_1)\Phi(\xi_2, \xi_3)$. The class of coordinate systems where an incomplete factorization is achievable obviously includes those for which a complete factorization is possible.

This Appendix will be limited to the discussion of the Helmholtz equation. This equation arises when the time dependence of the wave equation

$$\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = 0 \tag{A3.1}$$

is simple harmonic, so that the solution may be written as $\Psi = \psi(\mathbf{x})e^{i\omega t}$. Substitution into Eq. (A3.1) gives

$$\nabla^2\psi(\mathbf{x}) + k^2\psi(\mathbf{x}) = 0, \tag{A3.2}$$

where $k = \omega/c$. The Helmholtz equation results when the frequency ω is not a function of position.

The method of separation of variables is applicable to the Helmholtz equation in only eleven coordinate systems, generated by confocal quadrics or their degenerate forms.[3] The three families of confocal quadric surfaces are generated by the equation

$$\frac{x^2}{\zeta^2 - a^2} + \frac{y^2}{\zeta^2 - b^2} + \frac{z^2}{\zeta^2 - c^2} = 1, \qquad a \geq b \geq c \geq 0 \tag{A3.3}$$

by assigning different values to the parameter ζ: $\zeta > a$ generates a complete family of confocal ellipsoids; $a > \zeta > b$ generates a complete set of confocal hyperboloids of one sheet; and $b > \zeta > c$ generates a complete family of confocal hyperboloids of two sheets. The eleven coordinate systems correspond to the general ellipsoidal system and ten degenerate forms of the ellipsoidal coordinates obtained by setting a, b, and c equal to zero or infinity.

To determine the conditions that need to be satisfied for separability in an orthogonal coordinate system, the Helmholtz equation is written in terms of the curvilinear coordinates u^i as

$$\frac{1}{\sqrt{g}}\sum_{i=1}^{3}\frac{\partial}{\partial u^i}\left(\frac{\sqrt{g}}{g_{ii}}\frac{\partial\psi}{\partial u^i}\right) + k^2\psi = 0, \tag{A3.4}$$

where $g_{ij} = 0$ if $i \neq j$ (since the coordinates are orthogonal) and $g = g_{11}g_{22}g_{33} = h_1^2h_2^2h_3^2$. Assuming that the solution can be written as $\psi = U^1(u^1)U^2(u^2)U^3(u^3)$, substitution of this form into the last equation gives

$$\sum_{i=1}^{3}\frac{1}{U^i}\frac{\partial}{\partial u^i}\left(\frac{\sqrt{g}}{g_{ii}}\frac{\mathrm{d}U^i}{\mathrm{d}u^i}\right) + k^2\sqrt{g} = 0. \tag{A3.5}$$

If this equation is to separate, $\sqrt{g}/g_{ii}$ must be expressible as a product of two functions, one involving u^i, and the other independent of u^i; i.e.,

$$\begin{aligned}\frac{\sqrt{g}}{g_{11}} &= f_1(u^1)F_1(u^2, u^3),\\ \frac{\sqrt{g}}{g_{22}} &= f_2(u^2)F_2(u^1, u^3),\\ \frac{\sqrt{g}}{g_{33}} &= f_3(u^3)F_3(u^1, u^2).\end{aligned} \tag{A3.6}$$

If one assumes that this can be done, then Eq. (A3.5) can be written as

$$\sum_{i=1}^{3}\frac{F_i}{U^i}\frac{\mathrm{d}}{\mathrm{d}u^i}\left(f_i\frac{\mathrm{d}U^i}{\mathrm{d}u^i}\right) + k^2\sqrt{g} = 0. \tag{A3.7}$$

Equation (A3.7) is not yet separable because of the presence of F_i and $\sqrt{g}$ which are dependent on the coordinate system, but are independent of the boundary conditions. In general, the components U^i of the separated solution will depend on both the coordinate system and the three separation constants $\alpha_1 = k^2$, α_2, α_3, which in turn depend on the boundary conditions. To obtain the conditions required for separability, and there are two, one defines what is known as the Stäckel determinant. This is obtained by differentiating Eq. (A3.7) with respect to the separation constants α_i, thereby obtaining the three equations

$$\begin{aligned}f_1F_1\Phi_{11}(u^1) + f_2F_2\Phi_{21}(u^2) + f_3F_3\Phi_{31}(u^3) &= \sqrt{g},\\ f_1F_1\Phi_{12}(u^1) + f_2F_2\Phi_{22}(u^2) + f_3F_3\Phi_{32}(u^3) &= 0,\\ f_1F_1\Phi_{13}(u^1) + f_2F_2\Phi_{23}(u^2) + f_3F_3\Phi_{33}(u^3) &= 0,\end{aligned} \tag{A3.8}$$

where the quantities $\Phi_{ij}(u^i)$ are defined as

$$\Phi_{ij}(u^i) = -\frac{1}{f_i(u^i)}\frac{\partial}{\partial\alpha_j}\left[\frac{1}{U^i}\frac{\mathrm{d}}{\mathrm{d}u^i}\left(f_i\frac{\mathrm{d}U^i}{\mathrm{d}u^i}\right)\right], \qquad i, j = 1, 2, 3. \tag{A3.9}$$

Equations (A3.8) constitute a set of three simultaneous linear equations in the three unknowns $f_i F_i$. By Cramer's Rule, they have the solution

$$f_i F_i = \frac{\sqrt{g}}{S} M_{i1}, \tag{A3.10}$$

where S is the Stäckel determinant

$$S = \begin{vmatrix} \Phi_{11}(u^1) & \Phi_{12}(u^1) & \Phi_{13}(u^1) \\ \Phi_{21}(u^2) & \Phi_{22}(u^2) & \Phi_{23}(u^2) \\ \Phi_{31}(u^3) & \Phi_{32}(u^3) & \Phi_{33}(u^3) \end{vmatrix}, \tag{A3.11}$$

and M_{i1} is the cofactor of the element Φ_{i1} in the Stäckel determinant. Note that the rows and columns of the matrix of coefficients of Eqs. (A3.8) have been transposed in forming the determinant (the value of the determinant is unchanged by this operation), and for there to be a solution, S must not vanish. Each row of the Stäckel determinant is a function of only one of the curvilinear coordinates.

The correspondence between a Stäckel matrix and a coordinate system is not one-to-one. Two Stäckel matrices are equivalent if the ratio S/M_{i1}, the therefore the metric coefficients g_{ii}, are the same for each matrix. Equivalent Stäckel matrices are obtained by: (1) interchanging columns; (2) multiplication of each element of a column by a non-zero constant; and (3), adding one column to another. In addition, since S/M_{i1} is to be preserved, the first column has a privileged position and must not be altered.

Equation (A3.10) when combined with Eq. (A3.6) gives the first separability condition

$$g_{ii} = \frac{S}{M_{i1}}. \tag{A3.12}$$

When the functional dependencies of Eq. (A3.10) are explicitly written out as

$$\begin{aligned} \frac{\sqrt{g}}{S} &= f_1(u^1)\frac{F_1(u^2, u^3)}{M_{11}(u^2, u^3)}, \\ \frac{\sqrt{g}}{S} &= f_2(u^2)\frac{F_2(u^1, u^3)}{M_{21}(u^1, u^3)}, \\ \frac{\sqrt{g}}{S} &= f_3(u^3)\frac{F_3(u^1, u^2)}{M_{31}(u^1, u^2)}, \end{aligned} \tag{A3.13}$$

it is straightforward to show that the set can only be satisfied if

$$\frac{\sqrt{g}}{S} = f_1(u^1)f_2(u^2)f_3(u^3). \tag{A3.14}$$

This is known as the Robertson condition, and is the second required separability condition. Equations (A3.12) and (A3.14) are both necessary and sufficient conditions for simple separability of the Helmholtz equation.

Chapter 3 discussed the results of Weston[4] who showed that in cylindrical and rotational coordinate systems, one variable may be separated out of the Helmholtz equation, thereby leaving a second order partial differential equation in two variables. For a special class of coordinate systems, this second order equation is reducible to a recurrence set of ordinary differential equations in one variable.

The class of cylindrical and rotational coordinate systems used by Weston, and introduced by Moon and Spencer, can be introduced by means of conformal transformations. The reason for choosing such transformations is that they preserve angles. Recall that the only one-to-one mappings of a domain S in the $w = u + iv$ plane into a domain of the $z = x + iy$ plane (x and y being rectangular Cartesian coordinates) which preserves the angle of intersection between two curves is $z = \mathcal{F}(w)$ or $\bar{z} = \mathcal{F}(w)$, where $\mathcal{F}$ is an analytic function whose derivative is finite and non-zero at each point of S. If the sense of the angle is also conserved, the transformation is conformal. Since $\bar{z} = \mathcal{F}(w)$ is equivalent to $z = \mathcal{F}(w)$ followed by a reflection in the real axis of the z-plane, conformal transformations must be of the form $z = \mathcal{F}(w)$.

If the real and imaginary parts of $z = \mathcal{F}(w)$ are separated,

$$\begin{aligned} x &= \xi_1(u, v) \\ y &= \xi_2(u, v). \end{aligned} \tag{A3.15}$$

In the z-plane, these represent a family of curves u = constant and v = constant. Since the x and y axes are orthogonal, the family of curves u = constant and v = constant are also orthogonal under the conformal transformation $z = \mathcal{F}(w)$. If these curves are either translated along a line perpendicular to the plane containing them, or rotated, they will produce a triply orthogonal family of coordinate surfaces.

Cylindrical coordinates (u^1, u^2, u^3) are defined by a translation along the z-axis,

$$\begin{aligned} x &= \xi_1(u^1, u^2) \\ y &= \xi_2(u^1, u^2) \\ z &= u^3. \end{aligned} \tag{A3.16}$$

The coordinate surfaces are two mutually orthogonal families of cylinders and a family of parallel planes perpendicular to the z-axis; i.e., the z-axis is parallel to the generators of the cylinders.

Rotational coordinates are obtained by rotating the family of curves u = constant and v = constant about either the x or y axes. A rotation about the x-axis will rotate $\xi_2(u^1, u^2)$ about this axis, which is renamed the z-axis, so that

$$\begin{aligned} x &= \xi_2(u^1, u^2)\cos\phi \\ y &= \xi_2(u^1, u^2)\sin\phi \\ z &= \xi_1(u^1, u^2). \end{aligned} \tag{A3.17}$$

Similarly, one may rotate $\xi_1(u^1, u^2)$ about the y-axis (relabeling it the z-axis) to obtain

$$\begin{aligned} x &= \xi_1(u^1, u^2)\cos\phi \\ y &= \xi_1(u^1, u^2)\sin\phi \\ z &= \xi_2(u^1, u^2). \end{aligned} \tag{A3.18}$$

For both these cases, $u^3 = \phi$.

As a relevant example of this technique, the transformation

$$z = \frac{a\,(e^w + 1)}{e^w - 1}, \tag{A3.19}$$

can be used to generate toroidal coordinates which fall into the rotational class of coordinates. Here $z = x + iy$, and the definition of the real and imaginary parts of $w = \mu + i\eta$ have been relabeled so as to be consistent with the notation for toroidal coordinates used in Chapter 3. Equation (A3.19) can be expanded as

$$x + iy = \frac{a\,[e^\mu(\cos\eta + i\sin\eta) + 1]}{e^\mu(\cos\eta + i\sin\eta) - 1}. \tag{A3.20}$$

Multiplying both the numerator and denominator of the right hand side of this equation by the complex conjugate of the denominator gives

$$x + iy = \frac{a\,[(e^{2\mu} - 1) - 2ie^\mu\sin\eta]}{e^{2\mu} - 2e^\mu\cos\eta + 1}. \tag{A3.21}$$

x and y are then obtained by taking the real and imaginary parts, yielding

$$\begin{aligned} x &= \frac{a\,(e^{2\mu} - 1)}{e^{2\mu} - 2e^\mu\cos\eta + 1} = \frac{a\sinh\mu}{\cosh\mu - \cos\eta}, \\ y &= -\frac{2ae^\mu\sin\eta}{e^{2\mu} - 2e^\mu\cos\eta + 1} = -\frac{a\sin\eta}{\cosh\mu - \cos\eta}. \end{aligned} \tag{A3.22}$$

For consistency with the notation in Chapter 3, w should be defined as $w = \mu - i\eta$ rather than $w = \mu + i\eta$. This will change the sign of the $\sin\eta$ term. Assuming this to be the case,

and comparing with Eq. (A3.15), it is clear that $\xi_1(\mu, \eta)$ is given by the first of Eqs. (A3.22), and $\xi_2(\mu, \eta)$ by the second.

The toroidal coordinates of Chapter 3 are obtained by rotation about the y-axis, according to the prescription contained in Eqs. (A3.18),

$$\begin{aligned} x &= \frac{a\sinh\mu\cos\phi}{\cosh\mu - \cos\eta}, \\ y &= \frac{a\sinh\mu\sin\phi}{\cosh\mu - \cos\eta}, \\ z &= \frac{a\sin\eta}{\cosh\mu - \cos\eta}. \end{aligned} \tag{A3.23}$$

To understand why the Helmholtz equation will not separate in these coordinates requires an examination of the form of the Stäckel determinant for rotational coordinate systems. First, the metric coefficients may be obtained from Eqs. (A3.17) by using the fact that

$$g_{ii} = h_i^2 = \left|\frac{\partial \boldsymbol{r}}{\partial u^i}\right| = \left(\frac{\partial x}{\partial u^i}\right)^2 + \left(\frac{\partial y}{\partial u^i}\right)^2 + \left(\frac{\partial z}{\partial u^i}\right)^2, \quad i = 1, 2, 3. \tag{A3.24}$$

Thus,

$$\begin{aligned} g_{11} &= \left(\frac{\partial\xi_1(u^1, u^2)}{\partial u^1}\right)^2 + \left(\frac{\partial\xi_2(u^1, u^2)}{\partial u^1}\right)^2, \\ g_{22} &= \left(\frac{\partial\xi_1(u^1, u^2)}{\partial u^2}\right)^2 + \left(\frac{\partial\xi_2(u^1, u^2)}{\partial u^2}\right)^2, \\ g_{33} &= [\xi_2(u^1, u^2)]^2. \end{aligned} \tag{A3.25}$$

Since $\mathscr{A}$ is an analytic function, the Cauchy-Riemann conditions apply to Eq. (A3.15). This gives the relations:

$$\begin{aligned} \frac{\partial\xi_1(u^1, u^2)}{\partial u^1} &= \frac{\partial\xi_2(u^1, u^2)}{\partial u^2}; \\ \frac{\partial\xi_1(u^1, u^2)}{\partial u^2} &= -\frac{\partial\xi_2(u^1, u^2)}{\partial u^1}; \end{aligned} \tag{A3.26}$$

and, in particular,

$$\left(\frac{\partial\xi_1(u^1, u^2)}{\partial u^1}\right)^2 + \left(\frac{\partial\xi_2(u^1, u^2)}{\partial u^1}\right)^2 = \left(\frac{\partial\xi_1(u^1, u^2)}{\partial u^2}\right)^2 + \left(\frac{\partial\xi_2(u^1, u^2)}{\partial u^2}\right)^2; \tag{A3.27}$$

so that $g_{11} = g_{22}$. If Eq. (A3.18) is used rather than Eq. (A3.17), the result is

$$g_{11} = g_{22} = \left(\frac{\partial \xi_1(u^1, u^2)}{\partial u^1}\right)^2 + \left(\frac{\partial \xi_2(u^1, u^2)}{\partial u^1}\right)^2, \qquad g_{33} = [\xi_1(u^1, u^2)]^2. \tag{A3.28}$$

In general, for a coordinate system having axial symmetry, the metric coefficients contain only two variables and one may write

$$g_{11}(u^1, u^2) = g_{22}(u^1, u^2), \qquad g_{33} = g_{33}(u^1, u^2). \tag{A3.29}$$

This means that the third row of the Stäckel determinant, which is only a function of u^3, can only contain constants. It may then be written in the form

$$S = \begin{vmatrix} \Phi_{11}(u^1) & \Phi_{12}(u^1) & \Phi_{13}(u^1) \\ \Phi_{21}(u^2) & \Phi_{22}(u^2) & \Phi_{23}(u^2) \\ C_1 & C_2 & C_3 \end{vmatrix}. \tag{A3.30}$$

The first integrability condition, Eq. (A3.12), and the restriction on the metric coefficients given by Eq. (A3.29) imply that

$$\begin{aligned} g_{11}(u^1, u^2) = g_{22}(u^1, u^2) &= \frac{S}{M_{11}(u^2)} = \frac{S}{M_{21}(u^1)}, \\ g_{33}(u^1, u^2) &= \frac{S}{M_{31}(u^1, u^2)}. \end{aligned} \tag{A3.31}$$

This means that, for $S \neq 0$, $M_{11}(u^2) = M_{21}(u^1)$, which is only possible if these cofactors are equal to the same constant, which may be set equal to unity without loss of generality. Thus,

$$C_3\Phi_{22}(u^2) - C_2\Phi_{23}(u^2) = -C_3\Phi_{12}(u^1) + C_2\Phi_{13}(u^1) = 1. \tag{A3.32}$$

This condition may be satisfied by choosing $C_2 = 0$, and $\Phi_{22} = -\Phi_{12} = 1/C_3$. The Stäckel determinant then takes the form

$$S = \begin{vmatrix} \Phi_{11}(u^1) & -1/C_3 & \Phi_{13}(u^1) \\ \Phi_{21}(u^2) & 1/C_3 & \Phi_{23}(u^2) \\ C_1 & 0 & C_3 \end{vmatrix}. \tag{A3.33}$$

Since multiplication of each element of a column of a Stäckel determinant is an allowed operation, if the second column is multiplied by C_3, Eq. (A3.32) can only be satisfied if $C_3 = 1$. The determinant then has the form

$$S = \begin{vmatrix} \Phi_{11}(u^1) & -1 & \Phi_{13}(u^1) \\ \Phi_{21}(u^2) & 1 & \Phi_{23}(u^2) \\ C_1 & 0 & 1 \end{vmatrix}. \tag{A3.34}$$

From this form of the Stäckel determinant, the second of Eqs. (A3.31) can now be written as

$$g_{33}(u^1, u^2) = \frac{S}{M_{31}(u^1, u^2)} = -\frac{\Phi_{11}(u^1) + \Phi_{21}(u^2)}{\Phi_{23}(u^2) + \Phi_{13}(u^1)} + C_1. \tag{A3.35}$$

C_1 then amounts to a constant scale factor which, without loss of generality, can be set equal to zero. The final form of the Stäckel determinant for rotational coordinates is then

$$S = \begin{vmatrix} \Phi_{11}(u^1) & -1 & \Phi_{13}(u^1) \\ \Phi_{21}(u^2) & 1 & \Phi_{23}(u^2) \\ 0 & 0 & 1 \end{vmatrix}. \tag{A3.36}$$

The value of the determinant is

$$S = \Phi_{11}(u^1) + \Phi_{21}(u^2). \tag{A3.37}$$

Since $g_{11} = g_{22} = S$, the scale factors must be a sum of separated functions. This is also true for the ratio g_{11}/g_{33} since

$$\frac{g_{11}(u^1, u^2)}{g_{33}(u^1, u^2)} = \frac{S/M_{11}(u^1, u^2)}{S/M_{31}(u^1, u^2)} = -[\Phi_{23}(u^2) + \Phi_{13}(u^1)]. \tag{A3.38}$$

Since $\sqrt{g} = \sqrt{g_{11}g_{22}g_{33}} = \sqrt{S^2 g_{33}} = S\sqrt{g_{33}}$, Eq. (A3.14) can be used to write

$$\frac{\sqrt{g}}{S} = f_1(u^1)f_2(u^2) = \sqrt{g_{33}}, \tag{A3.39}$$

or,

$$g_{33}(u^1, u^2) = [f_1(u^1)f_2(u^2)]^2. \tag{A3.40}$$

The conditions on the g_{ii} given by Eqs. (A3.37), (A3.38), and (A3.40) are necessary and sufficient for simple separability of the Helmholtz equation. For toroidal coordinates, Eqs. (A3.23) and (A3.24) determine the g_{ii} to be

$$g_{11}(\mu, \eta) = g_{22}(\mu, \eta) = \frac{a^2}{(\cosh\mu - \cos\eta)^2},$$
$$g_{33}(\mu, \eta) = \frac{a^2 \sinh^2\mu}{(\cosh\mu - \cos\eta)^2}, \tag{A3.41}$$

as given in Eq. (3.133) of Chapter 3. These equations do not satisfy the condition of Eq. (A3.37) that requires that $g_{11}(\mu, \eta) = g_{22}(\mu, \eta)$ be a separable sum of functions of μ and η alone. Thus, the Helmholtz equation does not separate in these coordinates. Had the rotation in Eqs. (A3.22) been about the x-axis, rather than the y-axis, bispherical coordinates would have resulted. As is the case for toroidal coordinates, the metric coefficients $g_{11} = g_{22}$ for bispherical coordinates cannot be written as a separable sum; consequently, the Helmholtz equation will not separate in these coordinates either.

For completeness, the conditions needed for simple separation of the Helmholtz equation in the cylindrical coordinates of Eq. (A3.16) will also be derived, even though such coordinates did not play a role in Chapter 3. Equation (A3.24) determines the metric coefficients to be

$$g_{11} = \left(\frac{\partial\xi_1(u^1, u^2)}{\partial u^1}\right)^2 + \left(\frac{\partial\xi_2(u^1, u^2)}{\partial u^1}\right)^2,$$
$$g_{22} = \left(\frac{\partial\xi_1(u^1, u^2)}{\partial u^2}\right)^2 + \left(\frac{\partial\xi_2(u^1, u^2)}{\partial u^2}\right)^2, \tag{A3.42}$$
$$g_{33} = 1.$$

The Cauchy-Riemann conditions require that $g_{11} = g_{22}$, and these metric coefficients must equal $\sqrt{g}$.

The separability conditions of Eqs. (A3.12) and (A3.14) therefore require that

$$g_{11}(u^1, u^2) = g_{22}(u^1, u^2) = \frac{S}{M_{11}(u^2)} = \frac{S}{M_{21}(u^1)},$$
$$g_{33} = 1 = \frac{S}{M_{31}(u^1, u^2)} \tag{A3.43}$$
$$\frac{\sqrt{g}}{S} = \frac{g_{11}(u^1, u^2)}{S} = f_1(u^1)f_2(u^2),$$

where again the fact that the metric coefficients are not functions of u^3 implies that the third row of the Stäckel determinant can only contain constants. The second of Eqs. (A3.43) implies that $S = M_{31}(u^1, u^2)$. This means that $\Phi_{11} = \Phi_{21} = 0$, and $\Phi_{31} = 1$. The Stäckel determinant may then be written as

$$S = \begin{vmatrix} 0 & \Phi_{12}(u^1) & \Phi_{13}(u^1) \\ 0 & \Phi_{22}(u^2) & \Phi_{23}(u^2) \\ 1 & C_2 & C_3 \end{vmatrix}. \tag{A3.44}$$

Now the first of Eqs. (A3.43) can only be satisfied if $M_{11}(u^2) = M_{21}(u^1) = \text{Const.}$ This constant can again be chosen to be unity, resulting again in Eq. (A3.32). The fact that the allowable operations discussed after Eq. (A3.11) can be used to set $C_2 = 0$ (by multiplying the first column of Eq. (A3.44) by C_2 and subtracting from the second), suggests that Eq. (A3.32) can be satisfied by taking $C_2 = 0$, thereby requiring $C_3 = 1$ and $\Phi_{22}(u^2) = -\Phi_{12}(u^1) = 1$. The final form of the Stäckel determinant for cylindrical coordinates is then

$$S = \begin{vmatrix} 0 & -1 & \Phi_{13}(u^1) \\ 0 & 1 & \Phi_{23}(u^2) \\ 1 & 0 & 1 \end{vmatrix} = -\left[\Phi_{13}(u^1) + \Phi_{23}(u^2)\right]. \tag{A3.45}$$

Alternatively, one might have chosen $C_3 = 0$, and used the condition $M_{11}(u^2) = M_{21}(u^1) = 1$ to determine $C_2 = 1$, $\Phi_{23} = -1$, and $\Phi_{13} = 1$. The resulting form of the Stäckel determinant can still be brought into the form of Eq. (A3.45) by allowed operations.

Since $M_{11}(u^2) = M_{21}(u^1) = 1$, the first of Eqs. (A3.43) requires that

$$g_{11}(u^1, u^2) = g_{22}(u^1, u^2) = S(u^1, u^2), \tag{A3.46}$$

so that the third of Eqs. (A3.43) implies that $f_1(u^1) = f_2(u^2) = 1$. Thus, Eq. (A3.45) is the sole requirement for separability of the Helmholtz equation in cylindrical coordinate systems; i.e., $g_{11}(u^1, u^2) = g_{22}(u^1, u^2)$ must be expressible as a separated sum.

A3.2 Cyclides

Cyclides[5] are in general defined by the equation

$$u_0(x^2 + y^2 + z^2)^2 + u_1(x^2 + y^2 + z^2) + u_2 = 0, \tag{A3.47}$$

where u_0 is a constant, u_1 a polynomial of the first degree, and u_2 a polynomial of the second degree in x, y, z. If $u_0 \neq 0$, a general cyclide is then a surface of the fourth degree. Concentration here will be on the Dupin cyclides, which are special cases of this equation. Discussing the relationship between the general cyclide and the Dupin cyclides requires introducing the definition of singular points and minimum lines. Points where a unique

tangent line cannot be defined are known as singular points. Perhaps the simplest example is a curve that crosses itself. This example is quite general in the sense that it brings out the fact that any line through a singular point of a curve intersects the curve in at least two coincident points. A minimum line has the property that the distance between any two points on the line vanishes (recall that the coordinates are allowed to have imaginary values).

A Dupin cyclide, which is not a ring surface (torus), generally has four finite singular points, two of which are connected with the other two by minimum lines. It is possible that two of the singular points coincide. If this is the case, the cyclide has three finite singular points, two of which are connected with the third by minimum lines. The case of the ring cyclide will be discussed below.

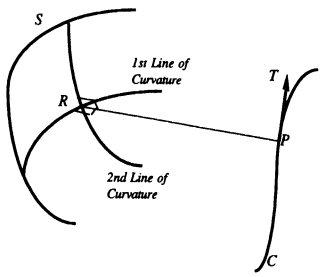


Fig. A3.1. Geometry used to determine the condition that all normals to the surface S pass through a fixed curve C.

Dupin's cyclides were defined by him in his *Applications de Géométrie* as the envelope of a family of spheres which are tangent to three fixed spheres. The name derives from the fact that the lines of curvature of the cyclides are always circles. Geometrically, the concept can be developed by considering Fig. A3.1. In the figure, R is a point on the surface S, and RP is a normal to the surface at R that meets a fixed curve C at P. The tangent to the curve at P is PT, and the points RPT determine a plane. The objective here is to determine the condition needed to ensure that all the normals of the surface pass through the fixed curve C.

The principal directions corresponding to the principal curvatures of any surface are orthogonal and determine the lines of curvature. These are indicated in the figure. If it is assumed that the first line of curvature is tangent to the plane RPT, the second will be perpendicular to it. If the plane RPT is now rotated taking the tangent PT as the axis of rotation, the second line of curvature will remain normal to the plane, and if the normal is to continue to pass through the fixed line, the second line of curvature must be a circle.

The normals to the surface S from the second line of curvature can be considered to be the radii of a sphere centered on P and tangent to the surface along the second line of curvature. The surface is then the envelope of a series of spheres whose centers lie on the fixed curve. The radii of the series may vary as P moves along the curve C.

If the normals to the surface are required to pass through two fixed curves rather than one, the surface will be the envelope of a second series of spheres whose centers lie on the second fixed curve, and each of which is tangent to all the spheres of the first series. Take any three spheres of the first series; the Dupin cyclides are then defined as the envelope of the series of spheres tangent to these three given spheres.

The two fixed curves, given the constraints above, must have very special properties, and it is readily proved that the curves are plane conics. One of the conics is the base of a right circular cone whose vertex is a point on the other. The planes containing the

conics are therefore at right angles to each other, and the foci of one are the vertices of the transverse axis of the other. These conditions are fulfilled, for example, by the ellipse and hyperbola given in parametric form by

$$x = c\cos\alpha, \quad y = \left(c^2 - b^2\right)^{1/2}\sin\alpha, \quad z = 0 \tag{A3.48}$$

and

$$x = b\sec\beta, \quad y = 0, \quad z = \left(c^2 - b^2\right)^{1/2}\tan\beta. \tag{A3.49}$$

Let P be a point on the ellipse, and Q a point on the hyperbola. Choose a point R between P and Q. If P (or Q) is fixed while Q (or P) varies, R will trace out a circle. The planes of the two circles (one obtained by fixing P and varying Q, and the other by fixing Q and varying P) are perpendicular, and the system of circles found in this way form a cyclide.

The distance between P and Q is, from Eqs. (A3.48) and (A3.49),

$$\begin{aligned} PQ &= \left[\left(c\cos\alpha - b\sec\beta\right)^2 + \left(c^2 - b^2\right)\sin^2\alpha + \left(c^2 - b^2\right)\tan^2\beta\right]^{1/2} \\ &= c\sec\beta - b\cos\alpha. \end{aligned} \tag{A3.50}$$

If P is fixed on the ellipse, $b\cos\alpha$ is a constant since α is fixed, and because R must trace out a circle as Q varies, the distance between P and R is fixed. The distance between R and Q can then be written as the distance between P and Q minus the fixed distance between P and R. If PR is written as a constant r minus the constant $b\cos\alpha$, RQ is

$$RQ = c\sec\beta - r. \tag{A3.51}$$

On the other hand, if Q is fixed on the hyperbola, by a similar argument,

$$PR = r - b\cos\alpha. \tag{A3.52}$$

The constant r must be the same in Eqs. (A3.51) and (A3.52) since $PQ = PR + RQ$.

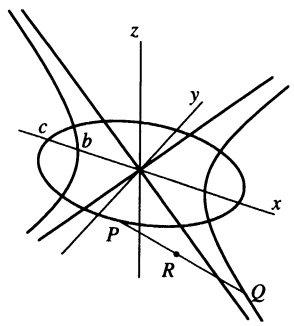


Fig. A3.2. The conics used to generate the cyclides. The foci of the ellipse and vertices of the hyperbola are at $x = \pm b$; the foci of the hyperbola and vertices of the ellipse are at $x = \pm c$. The coordinates of the point P located on the ellipse are $\{c\cos t, (c^2 - b^2)^{1/2}\sin t, 0\}$, while those of point Q on the hyperbola are given by $\{b\sec u, 0, (c^2 - b^2)^{1/2}\tan u\}$. The coordinates of R are $\{x, y, z\}$.

The different Dupin cyclides are obtained by choosing different values of r for fixed b and c. It will be shown shortly that there are four types of cyclides. Three of the these are: $0 < r < b$ the horned cyclide; $b < r < c$ the ring cyclide; and $r > c$ the spindle cyclide. The fourth type is the class of parabolic cyclides obtained by increasing the values of b, c, r, and x by a common value that is allowed to increase without limit. The two plane conics used to generate the cyclides become, in the limit, two parabolas in perpendicular planes, the focus of one being the vertex of the other. The relationship between r, b, and c again determine different forms of the parabolic cyclide. The arrangement of the conics used to generating the cyclides is shown in Fig. A3.2, while examples of the cyclides are shown in Figs. A3.3 through A3.6. These figures were generated by implementing the above scheme.

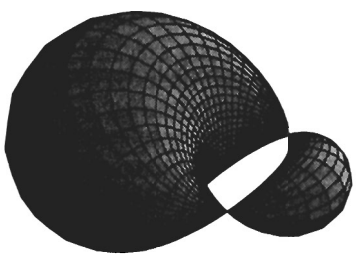
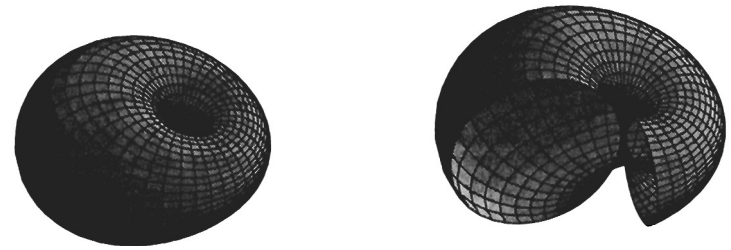


Fig. A3.3. A horned cyclide for which $r < b < c$. Each lobe terminates in two singular conical points.



(a) (b)

Fig. A3.4. A ring cyclide for which $b < r < c$. (a) shows the full surface and (b) a view of the interior.

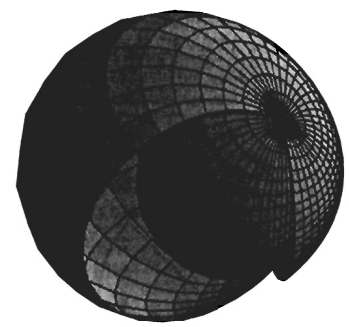


Fig. A3.5. A spindle cyclide where a portion of the outer surface has been cut away to show the interior sheet. The two sheets meet at two conical points. Here $r > c > b$.

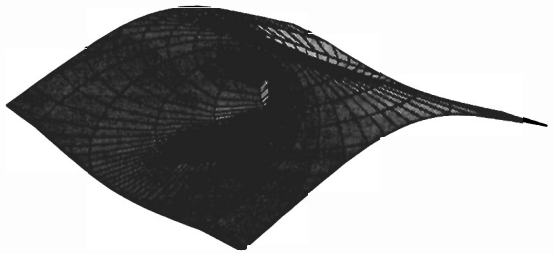


Fig. A3.6. A parabolic cyclide. Here $c > r > b$ and c, r, b, and the x-coordinate are increased by a large quantity. In the limit that this quantity becomes infinite, the generating conics become two parabolas in perpendicular planes, the focus of one being the vertex of the other. If r does not lie between b and c, the bridge becomes a lobe attached to the infinite sheet by conical points as in Fig. A3.3.

Study of the Dupin cyclides is greatly facilitated by use of the technique of transformation by reciprocal radius (inversion with respect to a sphere). With reference to Fig. A3.7, let $\mathcal{O}P' = R'$ and $\mathcal{O}P = R$. Then, if both sides of the inversion $R' = k^2/R$ are squared,

$$(x' - x_0)^2 + (y' - y_0)^2 + (z' - z_0)^2 = \frac{k^4}{R^4}\left[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2\right]. \quad \text{(A3.53)}$$

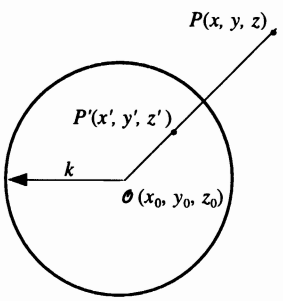


Fig. A3.7. $\mathcal{O}$ is the center of the sphere of inversion. The point P is transformed to P' under the transformation $\mathcal{O}P' = k^2/\mathcal{O}P$.

The basic equations for the inversion are then

$$\begin{aligned} x' &= x_0 + \frac{k^2}{R^2}(x - x_0), \\ y' &= y_0 + \frac{k^2}{R^2}(y - y_0), \\ z' &= z_0 + \frac{k^2}{R^2}(z - z_0). \end{aligned} \quad \text{(A3.54)}$$

Under an inversion, spheres are generally transformed into spheres with, however, some exceptions. The general equation of a sphere is given by

$$a(x^2 + y^2 + z^2) + bx + cy + dz + e = 0. \quad \text{(A3.55)}$$

Choosing the origin for the center of inversion, and applying the transformation of Eqs. (A3.54), Eq. (A3.55) becomes

$$ak^4 + bk^2x' + ck^2y' + dk^2z' + e\left(x'^2 + y'^2 + z'^2\right) = 0. \quad \text{(A3.56)}$$

Thus, spheres are transformed into spheres, except as follows: if $e = 0$ and $a \neq 0$, Eq. (A3.55) is a sphere through the center of inversion, and Eq. (A3.56) is a plane that does not pass through the center of inversion; if $e = 0$ and $a = 0$, Eqs. (A3.55) and (A3.56) are the same plane through the center of inversion; and if $e \neq 0$ and $a = 0$, Eq. (A3.55) is a plane that does not pass through the center of inversion, and Eq. (A3.56) is a sphere that passes through the center of inversion.

While it will not be proven here (the proof is straight forward), under an inversion angles between curves are conserved; i.e., the transformation is conformal. Consider now the three spheres of Fig. (A3.8). It is always possible to construct a circle, in the plane defined by the centers of the spheres, that is orthogonal to all three spheres. If any point on the circle is taken as the center of inversion, the circle is transformed into a straight line which is orthogonal to the three transformed spheres, and therefore passes through their centers. This is a consequence of the fact that the transformation is conformal.

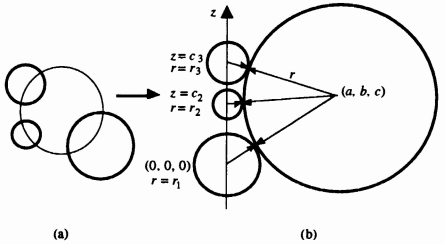


Fig. A3.8. Inversion of a circle perpendicular to three spheres into a straight line (the z-axis) orthogonal to the three transformed spheres. The center of the transformed sphere with radius r_1 is at the origin. The large circle with center (a, b, c) is constructed to be tangent to the three transformed spheres.

The Dupin cyclide, defined as the envelope of a family of spheres tangent to three fixed spheres is, under inversion, transformed into a surface that is the envelope of a family of spheres tangent to three spheres whose centers lie on a straight line. The equation of the three spheres with centers on the z-axis of Fig. A3.8(b) is

$$x^2 + y^2 + z^2 = r_1^2,$$
$$x^2 + y^2 + (z - c_2)^2 = r_2^2, \qquad \text{(A3.57)}$$
$$x^2 + y^2 + (z - c_3)^2 = r_3^2.$$

If the large sphere in Fig. A3.8(b), with center at (a, b, c), given by

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2, \qquad \text{(A3.58)}$$

is tangent to each of the three spheres on the z-axis, the distance between the centers of each of these spheres and the large sphere must be equal to either the sum or the difference of the radii (it is drawn as the sum). This means that

$$a^2 + b^2 + c^2 = (r \pm r_1)^2,$$
$$a^2 + b^2 + (c_2 - c)^2 = (r \pm r_2)^2, \qquad \text{(A3.59)}$$
$$a^2 + b^2 + (c_3 - c)^2 = (r \pm r_3)^2.$$

These are three equations in the four unknowns a, b, c, and r. By symmetry, however, $a^2 + b^2$ must be a constant. The general solution will then have the form c = constant, r = constant, and $a^2 + b^2$ = constant. There are four distinct cyclides under inversion determined by the condition that the surface be the envelope of a family of spheres tangent to three fixed spheres. In each case the envelope is a torus. The four different cyclides correspond to any one of the three fixed spheres being on the opposite side of the tangent sphere from the other two, or all being either inside or outside the tangent sphere. Thus, any Dupin cyclide is the inverse of a torus, that is a surface generated by revolving a circle about an axis in the plane of the circle. Note that the circle intersects the z-axis in two real, imaginary, or coincident points corresponding to whether the distance between the z-axis and the center of the circle is less, greater, or equal to the radius of the circle. These points are singular points, and represent the minimum number of singular points for a Dupin cyclide.

Since inversion is a conformal transformation, the orthogonal coordinates introduced on the torus in Section 4.1.1 are transformed into an orthogonal network on any of Dupin's cyclides. Further, since circles are transformed into circles under inversion, the orthogonal network on the cyclides consist of two families of circles. Also under inversion, spheres go into spheres so that any cyclide is inverted to another cyclide, the type depending on the location of the center of inversion.

Some additional observations on the general cyclides given by Eq. (A3.47) will be offered here without proof: (1) if the point of inversion is not on the cyclide, the inverse of any cyclide is always a cyclide of the fourth order; if the center of inversion is a non-singular point on the cyclide, the inverse is always a cyclide of the third order; (2) any cyclide with a singular point is the inverse of a quadric surface; conversely, the inverse of a quadric surface is a cyclide with at least one singular point; (3) any cyclide with two singular points not on the same minimum line is the inverse of a quadric cone; conversely, if the center of inversion is not on the cone, the inverse of a quadric cone is a cyclide with at least two singular points; (4) any cyclide of the fourth order with four singular points is a Dupin cyclide; and conversely, a Dupin cyclide of the fourth order generally has four singular points.

A3.3 Boundary Conditions for the Helmholtz and Vector Wave Equation

The Helmholtz equation is an elliptic partial differential equation. As such, the appropriate boundary conditions, depending on whether the boundary is open or closed, are shown in Table A3.1. In the table, Dirichlet boundary conditions means that $\psi(s)$ may be specified; Neumann conditions means that $N(s) = \hat{\boldsymbol{n}} \cdot \nabla \psi$ is given, where $\hat{\boldsymbol{n}}$ points inward toward the side of the boundary where the solution applies, and s is a point on the boundary; and Cauchy conditions means that both $\psi(s)$ and $N(s)$ are specified.

Boundary Conditions	Boundary Configuration	
	Open	Closed
Dirichlet	Non Unique	Unique, Stable Solution
Neumann	Non Unique	Unique, Stable Solution
Cauchy	Solution Unstable	Over Specified

As can be seen, a unique stable solution exists only for closed surfaces, and then only for Dirichlet or Neumann boundary conditions. When Neumann conditions are specified on a closed boundary, it may be necessary to satisfy an additional integral requirement over the boundary surface. For example, if $N(s)$ corresponds to the normal component of the velocity of an incompressible fluid, the integral condition would be the conservation of fluid mass. Also, since only the gradient is specified on the boundary, to obtain a unique solution it is necessary to specify ψ at one point of the boundary.

In the case of an open boundary configuration, for example the exterior of a sphere, unique and stable solutions can be obtained by imposing additional conditions at infinity. The Sommerfeld radiation condition

$$\lim_{r \to \infty} r\left(\frac{\partial \psi}{\partial r} - ik\psi\right) = 0, \qquad \lim_{r \to \infty} \psi = 0 \qquad \textbf{(A3.60)}$$

is a case in point. If the time dependence is put back into Eq. (A3.2), it can be seen that the radiation condition serves to exclude incoming, or converging waves from infinity. The Sommerfeld conditions are actually quite general in that, at a great distance from the sources, diverging waves from any configuration of sources will have the same features as diverging spherical waves.

It is also possible to impose mixed boundary conditions. Such conditions may be homogeneous,

$$a\psi(s) + bN(s) = 0, \tag{A3.61}$$

or inhomogeneous, where

$$a\psi(s) + bN(s) = F(s), \tag{A3.62}$$

and a and b can be specified independently of s. Homogeneous Dirichlet conditions ($b = 0$) means that $\psi(s)$ will be zero on the boundary; homogeneous Neumann conditions ($a = 0$) means that $N(s)$ will be zero on the boundary; and inhomogeneous Dirichlet or Neumann conditions means that $\psi(s)$ or $N(s)$ will have non-zero values specified by $F(s)$.

The approach often used to satisfy given boundary conditions is to make use of an expansion of the solution in terms of eigenfunctions. Boundary value problems may, however, be insoluble unless a coordinate system can be found such that the boundary surface is one of the coordinate surfaces. In addition, the equation to be solved must be separable in these coordinates if it is to be possible to find an appropriate set of eigenfunctions. The boundary conditions for eigenfunctions are always homogeneous, assuming the boundary is composed of ordinary points. In general, unlike the exterior problem for the sphere, the eigenvalues can be degenerate where each eigenvalue corresponds to several non-trivial independent solutions.

While satisfying given boundary conditions for the scalar Helmholtz equation may be difficult, the problem is far more arduous for the vector equation $\nabla^2\boldsymbol{B} + k^2\boldsymbol{B} = 0$. Even if it is possible to find a coordinate system appropriate for the boundary, and the components of $\boldsymbol{B}$ are expressed along the coordinate axes, the expression for ∇^2 in this coordinate system may mix the components of $\boldsymbol{B}$. That is, the three component equations for the vector Helmholtz equation do not necessarily separate into equations involving only one component of $\boldsymbol{B}$.

The boundary problem can be made more tractable by separating the vector field $\boldsymbol{B}$ into longitudinal and transverse components. The longitudinal component is written as the gradient of a scalar potential and the transverse component as the curl of a vector potential. That is, $\boldsymbol{B} = \boldsymbol{B}_L + \boldsymbol{B}_T$, where $\boldsymbol{B}_L = \nabla\varphi$ and $\boldsymbol{B}_T = \nabla\times\boldsymbol{A}$; $\boldsymbol{B}_L$ is a lamellar vector and $\boldsymbol{B}_T$ is solenoidal. The unique decomposition of a vector field in this way is always possible, and is the content of Helmholtz's theorem. The introduction of the vector potential does, however, lead to an arbitrariness in the gauge which, for convenience, will be chosen to be $\nabla\cdot\boldsymbol{A} = 0$. A gauge transformation allows results found in this gauge to be expressed in any other.

Assume that the Helmholtz equation is separable in the orthogonal coordinates ξ_1, ξ_2, ξ_3, and that the boundary corresponds to $\xi_1 = c$. The longitudinal part of the field is given by $\nabla\varphi$, and the boundary conditions can be applied according to the prescription given in the introductory paragraphs of this section. The transverse part of the field, given by $\nabla\times\boldsymbol{A}$, can be derived from two additional scalar fields ψ and χ. Three scalar fields are

not required, as would be the case in general, because the divergence condition, $\nabla\cdot\boldsymbol{A} = 0$, imposes a linear relationship between the three fields, which reduces the number of independent scalar fields to two. The scalar fields ψ and χ can be chosen so that $\nabla\psi$ is tangential to $\xi_1 = c$, and $\nabla\chi$ is parallel to the unit vector $\hat{\boldsymbol{e}}_1$ normal to $\xi_1 = c$ and tangent to ξ_1.

In Section 3.1.2 it was shown that if Ψ satisfied the Helmholtz equation $\nabla^2\Psi + \alpha^2\Psi = 0$, the set of vectors

$$\boldsymbol{L} = \nabla\Psi, \quad \boldsymbol{T} = \nabla\times(\boldsymbol{a}\Psi), \quad \text{and} \quad \boldsymbol{S} = \frac{1}{\alpha}\nabla\times\boldsymbol{T}, \tag{A3.63}$$

where $\boldsymbol{a}$ was any constant unit vector, constituted three independent solutions to the vector Helmholtz equation. Similarly, if φ, ψ, and χ satisfy the scalar Helmholtz equation with constant k^2 (rather than α^2), then the set of vectors

$$\boldsymbol{L} = \nabla\varphi, \quad \boldsymbol{T} = \nabla\times(\hat{\boldsymbol{e}}_1\psi), \quad \text{and} \quad \boldsymbol{S} = \frac{1}{k}\nabla\times[\nabla\times(\hat{\boldsymbol{e}}_1\chi)] \tag{A3.64}$$

will also satisfy the vector Helmholtz equation for certain coordinate systems. These systems include rectangular coordinates, where ξ_1 may be x, y, or z; circular, elliptic, and parabolic cylindrical coordinates, where ξ_1 corresponds to z; and spherical and conical coordinates, where ξ_1 corresponds to the radius r. For the latter two coordinate systems, ψ and χ can be written as $r\psi$ and $r\chi$.

The solutions to the vector Helmholtz equation given by Eqs. (A3.64) make it relatively easy to apply the appropriate boundary conditions. For the five other coordinate systems generated by the confocal quadrics or their degenerate forms, for which the Helmholtz equation can be solved by the technique of separation of variables, solving the boundary problem for a given set of boundary conditions is extremely difficult, if not impossible.

It may be helpful to point out that the proof that Eqs. (A3.64) satisfy the Helmholtz equation relies on the fact that one may interchange the order of application of the curl and Laplacian operators. That this is true can be seen by use of the vector relation

$$\nabla\times(\nabla\times\boldsymbol{u}) = \nabla(\nabla\cdot\boldsymbol{u}) - \nabla^2\boldsymbol{u}. \tag{A3.65}$$

First, let $\boldsymbol{u} = \nabla\times\boldsymbol{v}$. This results in $\nabla^2(\nabla\times\boldsymbol{v}) = -\nabla\times[\nabla\times(\nabla\times\boldsymbol{v})]$. Now let $\boldsymbol{u} = \boldsymbol{v}$ in Eq. (A3.65) and take the curl. This gives $\nabla\times\nabla^2\boldsymbol{v} = -\nabla\times[\nabla\times(\nabla\times\boldsymbol{v})]$. Subtracting the two results shows that $\nabla^2(\nabla\times\boldsymbol{v}) = \nabla\times\nabla^2\boldsymbol{v}$. The proof that Eqs. (A3.64) satisfy the Helmholtz equation is then done by direct substitution.

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INDEX

adjoint
  equation 58
  operator 88
Aharonov and Bohm effect 96
Alexander duality theorem 91
Alfvén speed 104
axial symmetry 140
Ballabh's generalization 21
Beltrami fields 4, 17, 125
Bernoulli's equation 25, 28, 34, 38, 46
Bessel's equation 18, 19, 36, 53
Birkhoff's ergodic theorem 76
Borromean rings 66
boundary conditions 29, 42, 133, 151
boundary group 90, 91
Călugăreanu invariant 2
Cauchy-Schwarz inequality 83
chiral media 119, 124
circular dichroism 119
circularly polarized waves 21, 119, 120, 127, 128
co-differential operator 87, 88
coherence length 109
confocal
  cyclides 42, 133
  quadrics 42, 134
conformal transformation 137, 149
constitutive relations 1, 119
Cowling's theorem 59, 60
critical magnetic field 104
crossover 65
cycle group 90, 91
cyclide
  confocal 20, 42, 133
  Dupin 143
  general 143
  generating conics 146
  horned 146
  parabolic 148
  ring 147
  spindle 147
De Rham's theorems 79, 87, 89, 91, 92, 97, 100
dynamical system 75
electromagnetic waves 1, 119
ergodic, theorem 76
Euler characteristic 59
flow 76
flux
  cancellation 105
  flow 109
  function 3, 25, 26
    helical 4
  line lattice 103, 109
fluxon 109

force-free fields
and aerodynamics 5
and astrophysics 6, 104
and elecromagnetic waves 1, 119
and hydrodynamics 4, 17, 26
and superconductors 110, 113
linear 104
non-linear 104
force-reduced magnets 14, 103
framed links 4
free energy 109
gamma function 50
gauge invariance, transformation 2, 64, 88, 96
Gegenbauer (ultraspherical) polynomials 31, 130
generating function 33, 114
Ginzburg-Landau equations 111, 114, 115
Grad-Shafranov equation 25, 28, 34
helicity
generalized 5
kink 65, 67, 68
linking 67, 68
non-localizable 69
of a solar prominence 106
relative 106
twist 65, 67, 68
Helmholtz equation 21, 42, 46, 134
high-temperature superconductors 104
Hill's spherical vortex 5
Hodge
star or duality operator 78, 87
decomposition theorem 88
Hölder inequality for integrals 83
homologous 90
homology
classes 90, 91
group 90, 91
modulo a boundary 92
of open arcs modulo a boundary 92
relative 92
homotopy class 97
Hopf
asymptotic invariant 75, 79
bundle 74
invariant 2, 72, 74, 75, 81
map 72, 73, 74
index
of a closed curve 59
of a singular point 59
integral surface 57, 58
integrating factor 59
inversion with respect to a sphere 148, 149
isothermal velocity of sound 104
Jacobi polynomials 29, 129
Klein bottle 60
Lagrangian multiplier 2, 84
Legendre
functions, associated 48, 49, 50
polynomials 130
linking number 65, 74, 75, 91
asymptotic 75, 77, 80, 81
Gauss 2, 70, 72
mean 78
London equations 114
Lorentz force 1
Lundquist solution 17, 37, 40, 41, 100
magnetic
axis 3
energy storage 14, 103
field energy 2, 83, 95
helicity 2, 64
surface 2, 3, 33, 59
Maxwell's equations 21, 120
Maxwell stress tensor 11, 13
metric coefficients 26, 44
metric indecomposability 76
minimum energy fields 3, 85
modified Bessel functions 113
n-chain, cycle, simplex 74
non-measurable 76
one-parameter mapping 75
one-point compactification 72
optically active media 21, 119, 124

order parameter 111
paramagnetic moment 109, 110
penetration depth 108
period 89, 92, 95
plasma beta 104
Poincaré-Hopf theorem 59, 60
post-flare magnetic configuration 108
potential fields 104
Poynting vector 119, 120
pressure equilibrium 6
principal stresses 15
radial pressure-balance equilibrium relation 114, 115
rationally independent 61
relaxation of a toroidal plasma 3
Riccati equation 122
Riccati-Bessel functions 29
rotational coordinate systems 42, 133, 137
Runge-Kutta method 63
semiflow 76
separation of variables 133
and boundary conditions 153
simply connected 60, 64, 70, 86, 88, 97, 108
singular points 60, 143, 150
solar flare 106
solar prominence 6, 104
Sommerfeld radiation condition 151
spheromak 5
square toroid 20, 35
Stäckel
determinant 135
matrix 136
standing waves 119
stereographic projection 72, 73
stream function 25, 26, 54
stress quadric 15
superconducting coils 103
superposition principle 1
surface integral 86, 108
Taylor conjecture 3, 108
topological
accessibility 3, 108
charge 112
invariants 3, 4
toroidal coordinates 20, 41, 43, 133, 139
toroidally curved cylindrical co-ordinates 52
torus
2-dimensional 3, 60, 61, 64, 93, 96, 98
3-dimensional 62
knot 61
trefoil knot 68
triangulation 59
Trkalian fields 4, 17
twist number 65, 67
type-I superconductors 109
type-II superconductors 103, 109
universal covering space 96, 97
vector potential 2, 64, 69, 70, 71, 87, 88, 100
as a local surface gradient 100
Virial of Claussius 9
virial theorem 9, 10, 103
vortex stability 114, 115
vorticity equation 5
wave equation 21
Whitehead link 66
winding number 97, 112
writhing number 65, 67