A class of cylindrically symmetric solutions to the force-free magnetic field equations with nonconstant α

Gerald E. Marsh
Argonne National Laboratory, 9700 South Cass Avenue, Argonne, Illinois 60439

(Received 17 May 1990; accepted for publication 25 June 1990)

The general approach to cylindrically symmetric force-free magnetic fields first introduced by Lüst and Schlüter [Z. Astrophys. **34**, 263 (1954)], is restricted to fields of the form $\mathbf{H} = [0, H_\phi(r), H_z(r)]$, and subsequently used to determine a set of solutions to the force-free field equations with nonconstant α. The first element of the set is the well-known constant α solution of Lundquist [Ark. Fys. **2**, 361 (1951)]. These solutions may have practical applications with respect to high-temperature superconductors.

I. INTRODUCTION

In past years, there has been great interest in force-reduced magnets for thermonuclear and energy storage applications where high magnetic fields are needed. It is, for example, possible to design a truly force-free coil surrounded by a force-bearing ring.[1] Today, there is renewed interest in force-reduced magnetic field configurations due to the recent discovery of high-temperature superconductors. Because the materials that exhibit such behavior tend to be quite brittle, it is important to reduce the forces they might experience if they are to be used for high-field applications. In addition, Furth[2] has raised the possibility that force-free field configurations may have the potential to raise the critical magnetic field and current-density limits in such superconductors.

For finite force-free magnetic field configurations the virial theorem, which can be used to relate the stored magnetic energy of the system to the integral of the trace of the stress tensor over the magnetic field volume, sets limits that must be obeyed in practice. While stresses may be eliminated in a given region, they cannot be cancelled everywhere. In general, the use of force-free field configurations allow the forces needed to balance the outward pressure of the magnetic field to be reduced in magnitude by spreading them out over a larger region. While the virial theorem does indeed set limits for the minimum amount of structure required for magnets made of materials having a given strength, force-free configurations nonetheless appear to have the potential to contribute to improved designs for large magnets.

The virial theorem originally evolved out of the kinetic theory of gases, and was first extended to include magnetic fields by Chandrasekhar and Fermi.[3] For a clear discussion of the application of the virial theorem, see the paper by Parker.[4]

II. FORCE-FREE MAGNETIC FIELDS

The possibility that cosmic magnetic fields might satisfy the condition that the magnetic field in some region is everywhere parallel to the direction of current flow,

$$(\nabla \times \mathbf{H}) \times \mathbf{H} = 0, \tag{1}$$

was apparently first considered by Lüst and Schlüter.[5] For the latter equation to hold, $\mathbf{H}$ must satisfy

$$\nabla \times \mathbf{H} = \alpha \mathbf{H}, \tag{2}$$

where α is in general a function of position. Since

$$\nabla \cdot \mathbf{H} = 0, \tag{3}$$

it follows that

$$\nabla \alpha \cdot \mathbf{H} = 0. \tag{4}$$

This means that α is a constant on any given field line, and if the field lines cover a surface, then α is a constant on that surface.

Chandrasekhar[6] obtained an explicit solution for the case where $\alpha = \text{const}$ and where $\mathbf{H}$ has axial symmetry, and Furth *et al.*[7] considered a more general case where α is a function of rH_ϕ. Furth *et al.* expressed the basic equations in terms of the vector potential, and for the case of a linear relationship between the azimuthal components of the current and vector potential (equivalent to setting α equal to a constant) obtained the solution

$$H_\phi = A \cos(kz) J_1[r(\alpha^2 - k^2)^{1/2}], \tag{5}$$

$$H_z = A[(\alpha^2 - k^2)^{1/2}/\alpha] \cos(kz) J_0[r(\alpha^2 - k^2)^{1/2}], \tag{6}$$

$$H_r = A(k/\alpha) \sin(kz) J_1[r(\alpha^2 - k^2)^{1/2}], \tag{7}$$

where A and k are constants. For $k = 0$, the radial component vanishes and this becomes the solution given by Lundquist,[8] which represents the first element of the set of solutions given below.

III. CYLINDRICALLY SYMMETRIC FIELDS

The approach to cylindrically symmetric fields given here is based on that introduced by Lüst and Schlüter[5] and used by Chandrasekhar.[6] The condition that none of the components of a magnetic field have an azimuthal dependence means that $\mathbf{H}$ can be written as a sum of a toroidal and a poloidal field:

$$\mathbf{H} = \hat{\mathbf{z}} \times \mathbf{r} T(r,z) + \nabla \times [\hat{\mathbf{z}} \times \mathbf{r} P(r,z)]. \tag{8}$$

Here the axis of symmetry is along $\hat{\mathbf{z}}$ and the functions T and P respectively generate the toroidal and poloidal fields. Expressing the first term on the right-hand side of this

 0021-8979/90/203818-04$03.00

equation in terms of the azimuthal coordinate $\hat{\phi}$, and expanding the triple vector product in the second term allows the general form for **H** to be written as

$$\mathbf{H}=-r\frac{\partial P(r,z)}{\partial z}\,\hat{\mathbf{r}}+rT(r,z)\,\hat{\boldsymbol{\phi}}+\frac{1}{r}\frac{\partial r^2P(r,z)}{\partial r}\,\hat{\mathbf{z}}. \tag{9}$$

Computing $\nabla\times\mathbf{H}$ and using $\nabla\times\mathbf{H}=\alpha\mathbf{H}$, the expressions for the $\hat{\mathbf{r}}$ and $\hat{\mathbf{z}}$ components combine to give

$$\alpha\nabla(r^2P)=\nabla(r^2T), \tag{10}$$

while the $\hat{\phi}$ component yields

$$\frac{\partial^2P}{\partial r^2}+\frac{3}{r}\frac{\partial P}{\partial r}+\frac{\partial^2P}{\partial z^2}=-\alpha T. \tag{11}$$

Now one can either use the component form of Eq. (10) or $\nabla\alpha\cdot\mathbf{H}=0$ with the general form of **H** given by Eq. (9) to obtain

$$\frac{\partial(r^2P)}{\partial r}\frac{\partial\alpha}{\partial z}-\frac{\partial\alpha}{\partial r}\frac{\partial(r^2P)}{\partial z}=0. \tag{12}$$

This is the same as

$$\nabla(r^2P)\times\nabla\alpha=0. \tag{13}$$

Thus, $\nabla\alpha$ and $\nabla(r^2P)$ are in the same direction, and $\alpha=\alpha(r^2P)$ is only a function of r^2P. By again considering the component form of Eq. (10), the relationship between α and r^2T can be written

$$r^2T=\int\alpha(r^2P)d(r^2P). \tag{14}$$

Thus far, the only constraint imposed on the force-free field equations has been that none of the components of the magnetic field have an azimuthal dependence. Consider now the case where $\mathbf{H}=[0,H_\phi(r),H_z(r)]$. For this form of **H**, the equations are completely solvable for $\alpha=$ const and solvable in terms of an arbitrary function of r if α is not a constant. In addition, it can be seen from Eq. (9) that for this case P and T are only functions of r. The basic equations, Eqs. (10), (11), and (12), then become

$$\alpha\frac{d(r^2P)}{dr}=\frac{d(r^2T)}{dr}, \tag{15}$$

$$\frac{d^2P}{dr^2}+\frac{3}{r}\frac{dP}{dr}=-\alpha T, \tag{16}$$

$$\frac{d(r^2P)}{dr}\frac{d\alpha}{dz}=0. \tag{17}$$

In the last equation, $d\alpha/dz=0$ since α is a function of r^2P and P is only a function of r.

IV. SOLUTIONS TO THE FORCE-FREE FIELD EQUATIONS

Consider first the case where $\alpha=$ const. Equation (14) is then immediately integrable to $rT=\alpha rP$. Substituting for rT in Eq. (16) yields

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d(rP)}{dr}\right)+\left(\alpha^2-\frac{1}{r^2}\right)rP=0, \tag{18}$$

which is Bessel's equation for rP. Apart from a constant, the solution for P and T is $P=J_1(\alpha r)/r$ and $T=\alpha P=\alpha J_1(\alpha r)/r$, respectively. Using this result in the general expression for the magnetic field, Eq. (9), gives

$$\mathbf{H}=AJ_1(\alpha r)\,\hat{\boldsymbol{\phi}}+AJ_0(\alpha r)\,\hat{\mathbf{z}}, \tag{19}$$

where A is an arbitrary constant. This is the well-known result obtained by Lundquist.[8]

In the case that α is not a constant, the expression for α given by Eq. (15), and the fact that

$$H_\phi=rT \text{ and } H_z=\frac{1}{r}\frac{d(r^2P)}{dr} \tag{20}$$

can be used to rewrite Eq. (16) as

$$\frac{1}{H_\phi}\frac{dH_z}{dr}+\frac{1}{H_z}\frac{1}{r}\frac{d(rH_\phi)}{dr}=0. \tag{21}$$

Building on the work of Alfvén,[9] who introduced a new variable $k=H_\phi/H_z$, Murty[10] obtained the differential equation

$$\frac{1}{H_z}\frac{dH_z}{dr}+\frac{1}{(1+k^2)}\frac{k}{r}\frac{d(kr)}{dr}=0. \tag{22}$$

Because k is the tangent of the angle between **H** and $\hat{\mathbf{z}}$, finding k and H_z solves the problem. By separating this equation in terms of an arbitrary function of r, Murty found that the solution could be written as

$$H_z=H_0\exp\left(-\int_0^r\Phi(r)dr\right) \tag{23}$$

and

$$k^2=2r^{-2}\exp\left(2\int^r\Phi(r)dr\right)\times\int_0^r\Phi(r)r^2\exp\left(-2\int^r\Phi(r)dr\right)dr. \tag{24}$$

A somewhat different approach will be used here. Instead of introducing the new variable k, first multiply Eq. (21) by $H_\phi H_z$ and then separate in terms of an arbitrary function to obtain

$$H_z\frac{dH_z}{dr}=-\varphi(r) \tag{25}$$

and

$$H_\phi\frac{1}{r}\frac{d(rH_\phi)}{dr}=\varphi(r). \tag{26}$$

The first equation can be immediately integrated,

$$H_z^2=-2\int\varphi(r)dr+C, \tag{27}$$

where C is an arbitrary constant of integration.

By defining a new variable, u, such that $H_\phi=u^{-1}$, the second equation becomes

$$\frac{du}{dr}-\frac{u}{r}+u^3\varphi(r)=0. \quad (28)$$

This is Bernoulli's equation. [The equation that results from separating Eq. (22) in terms of an arbitrary function $\Phi(r)$ also results in a form of Bernoulli's equation.] It is reducible to the standard form of a linear equation of the first order, and has the solution

$$H_\phi^2=Ar^{-2}+2r^{-2}\int\varphi(r)r^2\,dr, \quad (29)$$

where A is an arbitrary constant of integration which is henceforth set equal to zero to guarantee that H_ϕ is finite on the z axis (for the same reason, the constant associated with the integration on the right-hand side of the equation is also set equal to zero). The solution given here, and that of Murty, are clearly related by a simple transformation.

The problem has thus been reduced to finding those functions $\varphi(r)$ that correspond to physically interesting results.

V. A CLASS OF NONCONSTANT α SOLUTIONS

If Eqs. (27) and (29) are differentiated, and the components of **H** set equal to the constant α values in Eq. (19), it is immediately apparent that $\varphi(r)=J_0(r)J_1(r)$. This suggests that functions of the form

$$\varphi(r)=\sum_{n,m=0}^{\infty}a_{nm}C_n(r)D_m(r), \quad (30)$$

where C_n and D_m are cylinder functions, and the a_{nm} are appropriately chosen constants, may be of interest. Such functions lead to integrals of the form

$$\int C_n(r)D_m(r)dr \text{ and } \int r^2C_n(r)D_m(r)dr$$

when determining H_z and H_ϕ. While such integrals are in general quite difficult to evaluate, the following two expressions given by Luke[11] allow the integrals to be evaluated for special cases:

$$\int^z J_\mu(t)J_{\mu+1}(t)dt=\sum_{k=0}^{\infty}J^2_{\mu+k+1}(z), \quad (31)$$

$$(\rho+\mu+\nu)\int^z t^{\rho-1}C_\mu(t)D_\nu(t)dt$$
$$+(\rho-\mu-\nu-2)\int^z t^{\rho-1}C_{\mu+1}(t)D_{\nu+1}(t)dt$$
$$=z^\rho[C_\mu(z)D_\nu(z)+C_{\mu+1}(z)D_{\nu+1}(z)]. \quad (32)$$

Setting $\rho=3$ in the second equation gives

$$(3+\mu+\nu)\int^z t^2C_\mu(t)D_\nu(t)dt$$
$$+(1-\mu-\nu)\int^z t^2C_{\mu+1}(t)D_{\nu+1}(t)dt$$
$$=z^3[C_\mu(z)D_\nu(z)+C_{\mu+1}(z)D_{\nu+1}(z)]. \quad (33)$$

Restricting consideration henceforth to Bessel functions of the first kind, and of integer order (relabel μ by m and ν by n) with real arguments, there are two cases that *prima facie* appear to be promising: $n=m+1$ and $n=-m+1$.

A. Case 1: $n=m+1$

Substituting $n=m+1$ into Eq. (33), it can be seen that an appropriate choice for the arbitrary separation function in Eqs. (25) and (26) is

$$\varphi(r)=\tfrac{1}{2}[(m+2)J_m(r)J_{m+1}(r)-mJ_{m+1}(r)J_{m+2}(r)]. \quad (34)$$

The integral of Eq. (29) is straightforward and gives

$$H_\phi^2=(m+1)J^2_{m+1}(r). \quad (35)$$

The right-hand side is positive definite for all values of r, so that H_ϕ is well defined. Using Eq. (31), the integral of Eq. (27) is

$$H_z^2=-(m+2)\sum_{k=0}^{\infty}J^2_{m+k+1}(r)$$
$$+m\sum_{k=0}^{\infty}J^2_{m+k+2}(r)+C. \quad (36)$$

Using the relation

$$J_0^2(r)+2\sum_{k=1}^{\infty}J_k^2(r)=1, \quad (37)$$

H_z^2 can be written as

$$H_z^2=-1+J_0^2(r)+2\sum_{k=1}^{m}J_k^2(r)-mJ^2_{m+1}(r)+C. \quad (38)$$

If $m=0$, this solution will match the constant α solution of Lundquist if $C=1$. However, setting $C=1$ will allow H_z^2 to be negative for some values of r, thus making H_z pure imaginary for these values.

B. Case 2: $n=-m+1$

Proceeding in the same manner, substitution of $n=-m+1$ into Eq. (33) suggests that $\varphi(r)$ be chosen as

$$\varphi(r)=J_m(r)J_{m-1}(r). \quad (39)$$

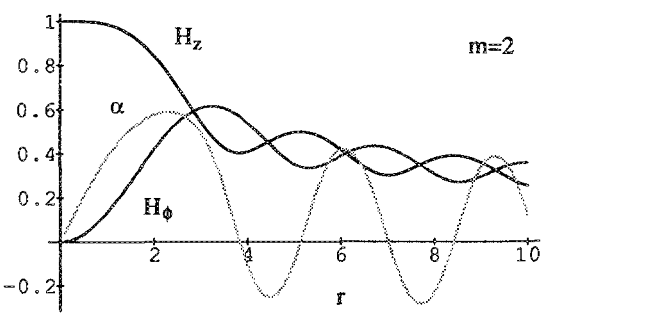

FIG. 1. Magnetic field components $H_z(r)$, $H_\phi(r)$, and the function $\alpha(r)$ for $m=2$.

The constant α case then corresponds to $m=1$. Substitution of Eq. (39) into Eqs. (27) and (29) yields, setting the constant of integration again equal to unity,

$$H_z^2=J_0^2(r), \quad m=1,$$

$$H_z^2=J_0^2(r)+2\sum_{k=1}^{m-1} J_k^2(r), \quad m>1, \tag{40}$$

and

$$H_\phi^2=(r/2)[J_m(r)J_{m-1}(r)-J_{m+1}(r)J_{m-2}(r)]. \tag{41}$$

Both Eqs. (40) and (41) are positive over the whole range of r, and therefore constitute a well-defined set of solutions to the force-free field equations that reduce to the Lundquist solution for $m=1$.

An expression for the function $\alpha(r)$ can readily be obtained from Eqs. (15) and (20),

$$\alpha(r)=\frac{1}{H_z}\frac{1}{r}\frac{d(rH_\phi)}{dr}. \tag{42}$$

Or, using Eqs. (27) and (29),

$$\alpha(r)=\frac{\varphi(r)}{H_zH_\phi}. \tag{43}$$

Substituting from Eqs. (39), (40), and (41)

$$\alpha(r)=\frac{J_m(r)J_{m-1}(r)}{\{(r/2)[J_m(r)J_{m-1}(r)-J_{m+1}(r)J_{m-2}(r)]\}^{1/2}\left(J_0^2(r)+2\sum_{k=1}^{m-1} J_k^2(r)\right)^{1/2}}, \quad m>1. \tag{44}$$

For $m=1$, Eq. (43) gives a constant value, $\alpha(r)=1$, as expected. The general behavior of $H_z(r)$, $H_\phi(r)$, and $\alpha(r)$ is shown in Fig. 1.

The angle between H and the z axis is given by $\tan^{-1}(H_\phi/H_z)$ and is shown in Fig. 2 . Note that the behavior shown in Fig. 2 is significantly different than that of the Lundquist solution given by $m=1$.

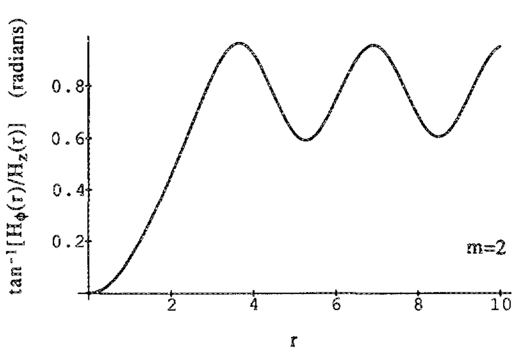

FIG. 2. Angle $\tan^{-1}[H_\phi(r)/H_z(r)]$ between H and the z axis for $m=2$.

VI. CONCLUSION

The general approach to force-free magnetic fields introduced by Lüst and Schlüter and used by Chandrasekhar, restricted to fields of the form $\mathbf{H}=[0,H_\phi(r),H_z(r)]$, was used to obtain a differential equation [Eq. (21)] that could be solved for H_ϕ and H_z in terms of an arbitrary separation function $\varphi(r)$. The form of this function needed to obtain the well known constant α solution of Lundquist then suggested a class of functions that result, for special cases, in integrable expressions for H_ϕ and H_z. Members of this class were then used to determine two sets of solutions to the force-free field equations with nonconstant α.

In practical applications, the various constants associated with the solutions would be chosen to both match required boundary conditions and ensure real values of the field over the range of r of interest.

[1] D. R. Wells and R. G. Mills, *High Magnetic Fields: Proceedings of the International Conference on High Magnetic Fields, November 1–4, 1961*, edited by H. Kolm, B. Lax, F. Bitter, and R. Mills (The MIT Press and Wiley, New York, 1962).
[2] H. P. Furth, S. C. Jardin, and D. B. Montgomery, IEEE Trans. Magn. **24**, 1467 (1988).
[3] S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 116 (1953).
[4] E. N. Parker, Phys. Rev. **109**, 1440 (1958). See also the discussion given by S. Chandrasekhar, in *Hydrodynamic and Hydromagnetic Stability* (Oxford, New York, 1961), p. 158.
[5] R. Lüst and A. Schlüter, Z. Astrophys. **34**, 263 (1954).
[6] S. Chandrasekhar, Proc. Natl. Acad. Sci. (USA) **42**, 1 (1956).
[7] H. P. Furth, M. A. Levine, and R. W. Waniek, Rev. Sci. Instrum. **28**, 949 (1957).
[8] S. Lundquist, Ark. Fys. **2**, 361 (1951).
[9] H. Alfvén, Ark. Fys. **19**, 375 (1961).
[10] G. S. Murty, Ark. Fys. **21**, 203 (1962).
[11] Y. L. Luke, *Integrals of Bessel Functions* (McGraw-Hill, New York, 1962). Integrals having the general form $\int f(x)C_\mu(x)D_\nu(x)dx$ have been treated by N. Sonine, Math. Ann. **16**, 1 (1880).