Although the gas-kinetic relation for thermal conductivity is indeed applicable, its validity under conditions quite different to those pertaining in a real gas should surely first be established in discussions of thermal conduction in nonmetals.

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[1]See, for example; C. Kittel, *Introduction to Solid State Physics* (Wiley, New York, 1971), 4th ed., pp. 224–225; J. M. Ziman, *Principle of the Theory of Solids* (Cambridge U. P., Cambridge, U. K.), 1st ed., p. 204.

[2]The phonon occupancy n_q of a mode of angular frequency ω_q is $n_q = [\exp(\hbar\omega_q/kT) - 1]^{-1}$. At high temperatures n_q approximates to $kT/\hbar\omega_q$. Summing over all distinct modes—the total number of modes is, of course, independent of T—shows that the total phonon population is proportional to T. At low temperatures the phonon occupancy of those modes for which $\omega_q > kT/\hbar$ is effectively zero. Adopting the Debye dispersion relation $\omega_q = qc$, where q is the circular wavenumber and c is the velocity of sound, we may therefore take it that those modes for which $q > kT/\hbar c$ are unoccupied while those modes up to $q = q_m = kT/\hbar c$ are occupied, each with an occupancy of $n_q = kT/\hbar\omega_q = kT/\hbar qc$. The total number of modes between q and $q + dq$ is proportional to $4\pi q^2 dq$. Since each has an occupancy $kT/\hbar qc$ the phonon population for such modes is proportional to $Tqdq$. Integrating from $q = 0$ to $q = q_m$ shows that the total phonon population n is proportional to T^3.

The mean phonon energy E is given by $E = \Sigma n_q E_q / \Sigma n_q = U/n$ where U is the internal energy of the solid. At high temperatures, where C ($= dU/dT$) is constant, $U \propto T$. Since $n \propto T$ it follows that E is constant (actually $E = k\theta_D$). At low temperatures $U \propto T^4$ in the T^3 heat capacity region and $n \propto T^3$, giving $E \propto T$ (actually $E = kT$).

[3]See, for example F. W. Sears, *Thermodynamics* (Addison–Wesley, Reading, MA, 1953), 2nd ed., pp. 269–273.

Does a gravitational red shift necessarily imply space–time curvature?

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Schild[1] has proposed a heuristic argument which attempts to show that any gravitational red shift requires that the geometry of space–time be curved. It is our intention to show that this argument is fallacious. A brief summary of the Schild argument is given below.

It is well known[2] that the principle of equivalence requires that given two clocks, one located at z_0 (gravitational potential V_0) and the other at z_1 (gravitational potential V_1, $V_1 > V_0$), then the clock at z_0 will run slower, the time difference being given by

$$(t_1 - t_0)/t_0 = V_1 - V_0 = \delta V \tag{1}$$

where we have set $c = 1$. (The above is true for any mass configuration and has been verified experimentally with an accuracy of about one percent.[3]) A photon is sent at time zero from z_0 to z_1. A time t_0 later a second photon is sent from z_0 to z_1. According to (1), the clock at z_1 will measure an elapsed time between the two photons of

$$t_1 = t_0(1 + \delta V). \tag{2}$$

The situation may be depicted by a Minkowski diagram with AB and CD being the world lines of the two photons, AC the world line of the clock at z_0, BD the world line of the clock at z_1. The situation being purely static, the two photon world lines are parallel, as are the two clock world lines. (Fig. 1.) Thus ABDC is a parallelogram and we should have

$$AC = BD.$$

On the other hand AC $= t_0$, BD $= t_1$, and from (2) $t_0 \neq t_1$. If ABDC is to be a parallelogram with AC $\neq$ BD, this can only be achieved if ABDC *is a parallelogram in non-Euclidian, curved, space–time*.

This completes our summary of the Schild argument. Most recently, Misner, Thorne, and Wheeler[4] have modified the argument by considering the photon world lines to be curved, because of the gravitational field, but still congruent, the clock world lines remaining straight and parallel to the T-axis. They again conclude that a "parallelogram" with unequal sides is obtained, which in turn mandates the conclusion that space–time is curved.

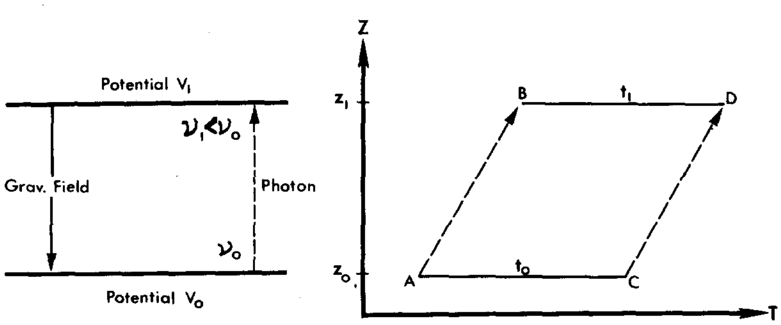

Fig. 1. Minkowski diagram as drawn by Schild[1] for two clocks in a coordinate system where both clocks are at rest. AC and BD are the world lines of the two clocks and AB and CD the world lines of the two photons sent from the clock at z_0 to the clock at z_1.

The Schild argument (and its Misner–Thorne–Wheeler version as well) can be applied to a well known specific situation where it predicts space–time curvature while in fact there is none:

Let the two clocks be in a uniform gravitational field with $V(z) = gz + V(0)$. The application of the Schild argument is straightforward and it predicts space–time curvature. On the other hand the Riemann curvature tensor vanishes everywhere and therefore so does the space-time curvature. Consequently, it is possible to find an inertial reference frame for which the *metric tensor assumes the diagonal Lorentz–Minkowski form (1, -1, -1, -1) everywhere.* In fact a freely-falling reference frame under a uniform gravitational field is such a frame.

The apparent paradox is readily resolved if we note that one cannot draw a *Z–T* Minkowski diagram as shown in Fig. 1. The frame in which the two clocks are at rest in the gravitational field is a *supported* frame and such a frame is not a proper Lorentz frame. A proper Minkowski diagram can be drawn only for a *freely falling* observer. For such an observer the world lines of the two clocks are not straight lines but rather hyperbolae, as shown in Fig. 2, and thus ABDC is not a parallelogram and there cannot be any expectation that AC = BD. A more detailed analysis[5] of the figure ABDC shows that indeed AB = CD and that the ratio of BD to AC is indeed $1 + gz$ as predicted by the principle of equivalence. (Incidentally, the supported observer himself can verify that there is no space–time curvature. He would obtain a negative result in any experiment testing geodesic deviation, for instance: "Do initially parallel light rays diverge?")

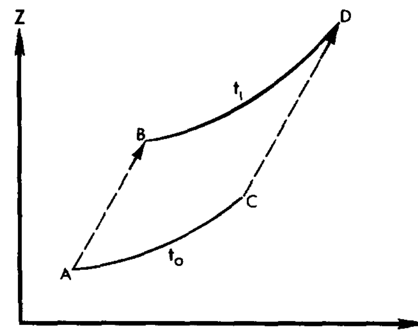

Fig. 2. Minkowski diagram for the two clocks as seen by a freely falling observer. AC and BD are the world lines of the two clocks and AB and CD the world lines of the two photons.

In conclusion we believe that no argument which attempts to infer space–time curvature solely from the gravitational red shift can be valid.

[1]A. Schild, Monist **47**, 20 (1962); *Evidence for Gravitational Theories*, edited by C. Møller (Academic, New York, 1962); *Conférence internationale sur les theories relativistes de la gravitation*, edited by L. Infeld (Pergamon, Oxford, 1964).

[2]H. Reichenbach, *The Philosophy of Space and Time* (Dover, New York, 1958).

[3]R. V. Pound and J. L. Snider, Phys. Rev. Lett. **13**, 539 (1964).

[4]C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).

[5]A. D. Fokker, *Time and Space, Weight and Inertia* (Pergamon, Oxford, 1965).

Motivating error analysis

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It has been our experience in teaching modern physics laboratory that students do not attach much importance to error analysis. Most students are indeed interested in obtaining the value of a fundamental constant or reproducing an important effect, but required error analysis is seen as a nuisance (on a level with correct spelling) and a sure sign of pedantry on the part of the instructor.

This is not really surprising in light of the intellectual framework of traditional laboratories. Students measure standard quantities which have been measured much more accurately by others. A student can determine the limitations of the equipment and techniques instantly by glancing at the textbook value. Why bother with a step-by-step error analysis which can only confirm the obvious?

In order to motivate students to treat error analysis seriously we have developed an error exam (game?) which requires the student to gamble for the grade he or she is to receive. Understanding the "odds" of this gamble requires understanding measurement errors.

The exam consists of a measurement of a single parameter whose value is unknown to the student but precisely set by the instructor. The answer to the exam is two numbers which the student believes bracket the true value. The grade received depends solely on these two numbers. Naturally the grade is higher the smaller the spread. But a miss gets a zero—no credit for effort, no partial credit, nothing. Consequently students are faced with a dilemma: How small can the uncertainty in a measurement be made without serious risk of missing the true value? The moment they realize the problem they have to solve, most students begin to get curious about errors and their meaning. (Descriptions of the exam are handed out a week ahead in order to give plenty of time for questions.)

The actual experiment we have used involves locating the height of a radioactive source hidden from view in a hollow cylinder. A sodium iodide detector can be moved along the outisde of the cylinder. If count rate is plotted versus position, a broad peak results. The centroid of the peak can be estimated and the location of the hidden source "determined." To obtain a standard deviation for the measurement, the students can repeat the process several times, or use a less accurate graphical technique familiar to them from a previous experiment.

The grading function we have used is designed so that anyone familiar with errors could expect at least a "B" after only a trivial analysis of the data. Yet about ten percent of the students fail the exam, usually because of a serious misconception about measurements or because of an oversight (systematic error).

Everyone seems to find the experiment challenging and amusing—amusing that is until the moment when the cylinder cover is to be removed and they are about to find out if their prediction is correct. Even students who lose the "bet" and fail the exam take the whole business very philosophically.

Some of the pedagogical techniques used here obviously could be adapted to other areas besides error analysis. In the general physics laboratory, to provide some excitement, a "prediction" could be added to almost any experiment. We give two examples:

(1) Following traditional air track experiments, an inclined air track could be set up and students required to predict the time for an air cart to pass between two photogates. At the end of the lab period, with predictions in hand, the instructor could turn on the equipment and measure the value with everyone watching.

(2) Following the determination of the velocity of a steel ball using a ballistic pendulum apparatus, students could predict the position on the floor where the launched ball would hit instead of simply comparing that value of the velocity with a second determination using the projectile method as in the traditional experiment.[1] We have tried this modification in a preliminary version and can report good student response.

[1] "Selective Experiments in Physics—Momentum: Ballistics." Central Scientific Co., 2600 South Kostner Avenue, Chicago, IL 60623.

A physical application of the function $\exp(-1/x^2)$

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It is frequently assumed that a convergent power series for a function converges to the function. The usual counterexample is $\exp(-1/x^2)$. The physicist may say, "But such functions never occur in physics." It is interesting, then, to see an actual physics problem whose answer depends on the behavior of this function for small x. This paper discusses the problem of a particle in a finite potential well in two dimensions, where the question to be answered is "Can there be a bound state for any depth of well, no matter how small?" We show that the solution depends on the exponential function previously mentioned.

It is generally assumed by physicists (and, fortunately, is usually true in physical problems) that the power series for a function represents the function within the circle of convergence; i.e., if the quantities $f^n(0)/n!$ can be computed and are finite then the power series so obtained converges to the function at all points where the series converges. The mathematician's counterexample is usually[1] $\exp(-1/x^2)$. This function, and all its derivatives, are zero at the origin, so the power series is simply zero. Yet it is clear that the values of the function itself, at points away from the origin, are not zero. The physicist's usual reply to being shown this function is, "Well, but such functions never occur in applications." It is of interest, then, to exhibit a rather straightforward example in quantum mechanics[2] where this function *does* arise and could give trouble.

The problem is as follows: It is known[3] that for a quantum-mechanical particle in a one-dimensional finite square well potential (i.e., $V = -V_0 < 0$, $|x| < a$, $V = 0$ for $|x| > a$) there is a bound state for any positive V_0a^2, and that in the analogous three-dimensional problem (i.e., in spherical coordinates, $V = -V_0 < 0$ for $r < a$, $V = 0$ for $r > a$) there is no bound state unless $V_0a^2 > \pi^2\hbar^2/8m$. (Note that we are assuming that in polar or spherical coordinates in two or three dimensions, r is automatically non-negative.) Question: What is the analogous situation for the two-dimensional case? (That is, in polar coordinates, $V = -V_0$ for $r < a$, $V = 0$ for $r > a$.) Note that this includes any (three-dimensional) problem with cylindrical symmetry, i.e., in which V is a function of the (cylindrical) coordinate r, only.

The radial equations for the two regions are

$$\frac{1}{r}(rR')' - \frac{1}{r^2}(n^2R) = -\alpha^2R, \quad r<a,$$

$$\frac{1}{r}(rR')' - \frac{1}{r^2}(n^2R) = \beta^2R, \quad r<a, \tag{1}$$

where

$$\alpha = [2m(V_0 - |E|)/\hbar^2]^{1/2},$$

$$\beta = (2m|E|/\hbar^2)^{1/2}. \tag{2}$$

Inside the well $(r < a)$ the solutions are

$$R = J_n(\alpha r) \tag{3}$$

(we must use only the J_n solutions because the N_n solutions are unbounded at the origin). Outside the well $(r > a)$ the solution is

$$R = K_n(\beta r), \tag{4}$$

where K_n is the hyperbolic Bessel function which tends to