Uniform magnetic fields in deflection coil design and the problem of the ellipsoid*

G. E. Marsh

Department of Biophysics and Theoretical Biology, University of Chicago, Chicago, Illinois 60637
(Received 19 February 1975)

The problem of obtaining a uniform magnetic field within a nondegenerate ellipsoid by the use of an appropriate surface current distribution is investigated. The curves of constant current on the surface of the ellipsoid are determined. The relation of the present work to the design of deflection coils is discussed.

PACS numbers: 03.50.C, 07.65.

I. INTRODUCTION

Methods of producing homogeneous magnetic fields within a given volume have a long history.[1,2] Tseitlin[3] has treated the problem for the case of an ellipsoid of revolution with the additional requirement that the external field vanish, and utilizing the hydrodynamic analogy[4] he has discussed the problem for an arbitrary closed volume. Laslett[5] has approached the problem, also with the requirement that the external field vanish, utilizing the well-known relation[6]

$$\mathbf{A}(x',y',z')=\frac{\mu}{4\pi}\int_V \frac{\mathbf{J}(x,y,z)}{r}\,dV-\frac{1}{4\pi}\int_S \frac{\hat{n}\times\mathbf{B}}{r}\,dS -\frac{1}{4\pi}\int_S(\hat{n}\times\mathbf{A})\times\nabla\left(\frac{1}{r}\right)dS -\frac{1}{4\pi}\int_S(\hat{n}\cdot\mathbf{A})\nabla\left(\frac{1}{r}\right)dS$$

(conventional symbols will be used throughout this paper) and applying a gauge transformation to eliminate the last term with its physically obscure interpretation.

The use of current sheets to produce bounded fields has also been treated by Richardson[7] in the cases where the boundary condition

$$\hat{n}\times(\mathbf{H}^+-\mathbf{H}^-)=\mathbf{j}$$

can be simplified by having $\mathbf{H}^+=0$ or normal to the sheet (e.g., when the sheet is shrouded in a material such as iron having a large permeability).

The purpose here is to investigate the problem of the distribution of windings needed to obtain a uniform field within an ellipsoid without the additional requirement of having the field vanish outside. This problem is known to be soluble for ellipsoids of revolution (spheroids) but has not been fully treated for general ellipsoids.

Section II introduces a "current function" Φ, and a specific form for this function is chosen. In Sec. III the components of the vector potential are found satisfying in addition to Laplace's equation the boundary conditions with the surface currents given by the specific choice of Φ. It is shown that the vector potential thus determined gives a uniform field within the ellipsoid. In Sec. IV the form of the curves of constant current on the surface of the ellipsoid is investigated, and Sec. V, in addition to containing some practical considerations, relates the present work to existing practice in the design of deflection coils.

II. CURRENT FUNCTION Φ

In order to motivate the introduction of Φ, consider first the example of a plane current sheet. The equation of continuity becomes

$$\nabla\cdot\mathbf{j}=\frac{du}{dx}+\frac{dv}{dy}=0, \tag{2.1}$$

where the surface current $\mathbf{j}=(u,v)$. This is the condition that the differential $u\,dy-v\,dx$ should be *exact*. There exists therefore a function Φ such that

$$u=\frac{d\Phi}{dy},\quad v=-\frac{d\Phi}{dx}. \tag{2.2}$$

The function Φ so introduced is called the current function.

In order to see how this can be generalized to three dimensions, one writes the surface current as $\mathbf{j}=(u,v,0)$. Since $\nabla\cdot\mathbf{j}=0$, $\mathbf{j}$ may be written as the curl of some vector field $\mathbf{a}$. As we expect the curl of $\mathbf{a}$ to be in the X-Y plane, $\mathbf{a}$ must be in the direction of the normal which is here $\hat{z}$. Then $\mathbf{a}=\Phi\hat{n}=(0,0,\Phi)$. Taking the curl then gives Eqs. (2.2).

In general if $\nabla\cdot\mathbf{j}=0$ on some surface, we may write $\mathbf{j}$ as the curl of some vector field $\mathbf{a}$. Letting $\mathbf{a}=\Phi\hat{n}$ we have $\mathbf{j}=\nabla\times\Phi\hat{n}=\nabla\Phi\times\hat{n}+\Phi\nabla\times\hat{n}$. For a closed surface given by $F=0$ the normal vector is $\hat{n}=\nabla F/|\nabla F|$, so that the current lines will be the closed curves $\Phi=\text{const}$, and $\mathbf{j}$ will be given by

$$\mathbf{j}=\nabla\Phi\times\hat{n}. \tag{2.3}$$

For an ellipsoid we have

$$F\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0, \tag{2.4}$$

so that the normal vector is

$$\hat{n}=\frac{\nabla F}{|\nabla F|}=\frac{2}{|\nabla F|}\left(\frac{x}{a^2}\,\hat{i}+\frac{y}{b^2}\,\hat{j}+\frac{z}{c^2}\,\hat{k}\right), \tag{2.5}$$

where

$$|\nabla F|=2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{1/2}.$$

We now choose the current function to be $\Phi=-Kz$. The components of the surface current are then

$$\bar{j}_x=\frac{2Ky}{b^2|\nabla F|},\quad \bar{j}_y=-\frac{2Kx}{a^2|\nabla F|},\quad \bar{j}_z=0 \tag{2.6}$$

III. DEMONSTRATION THAT $\Phi = -Kz$ YIELDS A UNIFORM INTERIOR FIELD

In magnetostatics surface currents normally arise as a result of volume distributions of current, and it is therefore conventional to emphasize the magnetization vector **M**. This has the unfortunate effect of obscuring the analogy which exists between electrostatics and magnetostatics. Here we write the boundary conditions for the vector potential explicitly in terms of the surface currents:

$$\frac{\partial \mathbf{A}^+}{\partial n} - \frac{\partial \mathbf{A}^-}{\partial n} = -\mu_0 \mathbf{j}. \tag{3.1}$$

The superscripts + and − designate **A** just outside and just inside the ellipsoid, respectively. For convenience, (3.1) in terms of components is written

$$\frac{\partial A_i^+}{\partial n} - \frac{\partial A_i^-}{\partial n} = -\mu_0 j_i, \quad i = x, y, z. \tag{3.2}$$

In addition to (3.2) the A_i must satisfy Laplace's equation $\nabla^2 A_i = 0$, which is here most easily solved in ellipsoidal coordinates. These are specified by

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} - 1 = 0, \quad \theta = \lambda, \eta, \zeta \tag{3.3}$$

which defines a triple orthogonal system of confocal quadrics:

$\lambda > -c^2$ ellipsoids,

$-c^2 > \eta > -b^2$ hyperboloid of one sheet,

$-b^2 > \zeta > -a^2$ hyperboloid of two sheets.

In terms of these coordinates, Laplace's equation is expressible as

$$(\eta - \zeta) R_\lambda \frac{\partial}{\partial \lambda}\left(R_\lambda \frac{\partial \phi}{\partial \lambda}\right) + (\zeta - \lambda) R_\eta \frac{\partial}{\partial \eta}\left(R_\eta \frac{\partial \phi}{\partial \eta}\right) + (\lambda - \eta) R_\zeta \frac{\partial}{\partial \zeta}\left(R_\zeta \frac{\partial \phi}{\partial \zeta}\right) = 0, \tag{3.4}$$

where

$$R_\theta = [(a^2+\theta)(b^2+\theta)(c^2+\theta)]^{1/2}. \tag{3.5}$$

If the family of confocal ellipsoids parameterized by λ are to be equipotentials we must have $\phi = \phi(\lambda)$ and (3.4) becomes

$$\frac{\partial}{\partial \lambda}\left(R_\lambda \frac{\partial \phi}{\partial \lambda}\right) = 0. \tag{3.6}$$

Integrating,

$$R_\lambda \frac{d\phi}{d\lambda} = \text{const};$$

so that

$$\phi = \text{const} \int_\lambda^\infty d\lambda / R_\lambda, \tag{3.7}$$

which is an elliptic integral of the first kind. Considering the boundary conditions (3.2) and the form of the surface currents (2.6) we want to consider solutions to (3.6), for example for A_y, of the form $A_y = xX$ where $X = X(\lambda)$. Laplace's equation $\nabla^2 A_y = 0$ becomes

$$\left(R_\lambda \frac{d}{d\lambda}\right)^2 X = -(b^2+\lambda)(c^2+\lambda)\frac{dX}{d\lambda}, \tag{3.8}$$

which can be written

$$\frac{d}{d\lambda} \log\left(R_\lambda \frac{dX}{d\lambda}\right) = -\frac{1}{a^2+\lambda},$$

from which

$$X = \text{const} \int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)R_\lambda}. \tag{3.9}$$

This is an elliptic integral of the second kind. Note that (3.9) can be written

$$X = -2\frac{d\phi}{da^2}, \tag{3.10}$$

where ϕ is given by (3.7). Define

$$\phi_0 \equiv \int_0^\infty d\lambda / R_\lambda. \tag{3.11}$$

The conditions on A_y can be satisfied by taking

$$A_y^+ = C_0 x \left(\frac{d\phi}{da^2}\right)\left(\frac{d\phi_0}{da^2}\right)^{-1}, \quad A_y^- = C_0 x, \tag{3.12}$$

where C_0 is a constant to be determined. Note that A_y^+ reduces to $C_0 x$ on the surface of the ellipsoid where $\lambda = 0$. The boundary conditions (3.2) may now be used to determine C_0. First

$$\begin{aligned}\frac{dA_y^+}{dn} &= \hat{n}\cdot\nabla A_y^+ \\ &= C_0\left\{\left[\left(\frac{d\phi}{da^2}\right)\left(\frac{d\phi_0}{da^2}\right)^{-1}\right]_{\lambda=0} \hat{n}\cdot\nabla(x) + x\hat{n}\cdot\nabla\left(\frac{d\phi}{da^2}\right)\left(\frac{d\phi_0}{da^2}\right)^{-1}\right\} \\ &= \frac{2C_0}{|\nabla F|}\left[\frac{x}{a^2} + x\left(\frac{d\phi_0}{da^2}\right)^{-1}\left(\frac{x}{a^2}\frac{d\lambda}{dx} + \frac{y}{b^2}\frac{d\lambda}{dy} + \frac{z}{c^2}\frac{d\lambda}{dz}\right)\frac{d}{d\lambda}\frac{d\phi}{da^2}\right].\end{aligned} \tag{3.13}$$

Now for $\lambda = 0$

$$\frac{d}{d\lambda}\frac{d\phi}{da^2} = -\frac{1}{2}\frac{d}{d\lambda}\int_\lambda^\infty \frac{d\lambda}{(a^2+\lambda)R_\lambda} = \frac{1}{2}\frac{1}{a^3bc}. \tag{3.14}$$

Furthermore it can be shown from (3.3) with $\lambda = 0$ that

$$\frac{d\lambda}{dx} = \frac{8x}{a^2}\frac{1}{|\nabla F|^2}, \quad \frac{d\lambda}{dy} = \frac{8y}{b^2}\frac{1}{|\nabla F|^2}, \quad \frac{d\lambda}{dz} = \frac{8z}{c^2}\frac{1}{|\nabla F|^2}. \tag{3.15}$$

Equations (3.13)–(3.15) together yield

$$\frac{dA_y^+}{dn} = \frac{2C_0}{|\nabla F|}\frac{x}{a^2}\left\{1 + \left[abc\left(\frac{d\phi_0}{da^2}\right)\right]^{-1}\right\}. \tag{3.16}$$

From (3.12),

$$\frac{dA_y^-}{dn} = \frac{2C_0}{|\nabla F|}\frac{x}{a^2}. \tag{3.17}$$

Applying the boundary conditions (3.2) to (3.16) and (3.17), we have

$$\begin{aligned}\frac{dA_y^+}{dn} - \frac{dA_y^-}{dn} &= \frac{2C_0}{|\nabla F|}\frac{x}{a^2}\left[abc\left(\frac{d\phi_0}{da^2}\right)\right]^{-1} = -\mu_0 j_y \\ &= \mu_0(2Kx/a^2|\nabla F|).\end{aligned} \tag{3.18}$$

The constant C_0 is then

$$C_0 = \mu_0 Kabc\left(\frac{d\phi_0}{da^2}\right) \tag{3.19}$$

and A_y^- is therefore given by

$$A_y^- = \mu_0 Kabc\left(\frac{d\phi_0}{da^2}\right)x. \tag{3.20}$$

In an exactly similar manner A_x^- is found to be

$$A_x^- = -\mu_0 Kabc\left(\frac{d\phi_0}{db^2}\right)y. \qquad (3.21)$$

and A_z^- is zero by the third of Eqs. (2.6). The field inside the ellipsoid is then given by $\mathbf{B}=\nabla\times\mathbf{A}$ which from (3.20) and (3.21) is

$$\mathbf{B}=\mu_0 abcK\left(\frac{d\phi_0}{da^2}+\frac{d\phi_0}{db^2}\right)\hat{k}, \qquad (3.22)$$

so that the internal field is seen to be uniform and in the z direction.

IV. CURVES OF CONSTANT CURRENT

The choice of $\Phi=-Kz$ for the current function and (2.3) tell us that the curves of constant current are such that $\hat{n}\times\hat{k}=\text{const}$. This is equivalent to requiring that $\hat{n}\cdot\hat{k}=\text{const}\equiv d$. The variation of d in the range $0\leqslant d\leqslant 1$ gives the set of curves of constant current covering the ellipsoid. From (2.5),

$$\hat{n}\cdot\hat{k}=\frac{z}{c^2}\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)^{-1/2}=d. \qquad (4.1)$$

This may be written

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}-\left(\frac{1}{d^2}-1\right)\frac{z^2}{c^4}=0, \qquad (4.2)$$

which is the equation of a real quadric cone. The intersection of this cone with the original ellipsoid (2.4) gives the required curve, which in general is of the fourth order. Instead of solving for its equation we use (2.4) to eliminate explicit reference to z:

$$\left(\frac{c^2}{a^2}\frac{d^2}{1-d^2}+1\right)\frac{x^2}{a^2}+\left(\frac{c^2}{b^2}\frac{d^2}{1-d^2}+1\right)\frac{y^2}{b^2}=1. \qquad (4.3)$$

This is the equation of an ellipse with semiaxes

$$a\left(\frac{c^2}{a^2}\frac{d^2}{1-d^2}+1\right)^{-1/2} \quad \text{and} \quad b\left(\frac{c^2}{b^2}\frac{d^2}{1-d^2}+1\right)^{-1/2};$$

i.e., the projection on the x-y plane of the curve of intersection of the quadric cone with the ellipsoid is an ellipse with the given semiaxes. With this in mind the practical problem of distributing windings on the ellipsoid becomes tractable.

The investigation of the z variation of the windings is facilitated by introducing $\hat{n}\cdot\hat{k}=\cos\theta$ and the parametrization of the ellipsoid

$$x^2=a^2\sin^2\alpha\cos^2\beta,$$
$$y^2=b^2\sin^2\alpha\sin^2\beta,$$
$$z^2=c^2\cos^2\alpha. \qquad (4.4)$$

With (4.1) this yields

$$\cot^2\alpha=c^2\cot^2\theta\left[\frac{1}{a^2}+\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin^2\beta\right]. \qquad (4.5)$$

Note that (4.3) and (4.4) immediately give (4.5). For $a=b$ the z variation vanishes. This corresponds to the case of a spheroid where the internal field will be uniform if

$$I\propto\sin\theta=[1+(a^2/c^2)\cot^2\alpha]^{-1/2}, \qquad (4.6)$$

which is equivalent, in terms of windings, to having a constant number of amp-turns per unit length along the z axis.[8] If in addition $a=c$ we have a sphere where $\alpha=\theta$.

V. SOME PRACTICAL CONSIDERATIONS

This work is of practical use, for example, in β-ray spectroscopy[9] and deflection coil design. In the latter case current practice[10] makes use of windings of the distributed type utilizing what is known as a sine distribution. In both these applications, and we will discuss only the second, it is advantageous to produce as uniform a magnetic field as possible. This is normally achieved by the use of Helmholtz coils or coils having "saddle" shapes. In the latter case the optimum geometry has been determined by Ginsberg and Melchner.[11] The use of Helmholtz coils suffers from two drawbacks: First it is difficult, if not impossible in most cases, to maintain the appropriate spacing for the coils in applications where both X and Y deflections are required. Second, both saddle-shaped coils and Helmholtz coils give only a small usable volume of uniform field[12] (typically 0.2 cubic radii for a 1% deviation).

These problems are readily overcome using the distributed windings referred to earlier. In terms of Sec. II such windings follow the curves $\Phi=\text{const}$ and the magnitude of the current is reflected by the density of windings. The sine distribution of Ref. 10 is obtained by distributing saddle-shaped coils over a cylinder. This corresponds to setting $a=\infty$, $b=c$, $\beta=\frac{1}{2}\pi$ in Eq. (4.5), giving $\alpha=\theta$. The distribution of windings is then such that the number of turns per angular element at the circumference of the circular cross section is proportional to $\sin\theta$. (The range of θ is $0\leqslant\theta\leqslant\pi$.)

Normally two windings arranged $\frac{1}{2}\pi$ apart are required to obtain both X and Y deflection. If both are distributed windings having sine distributions, we have

$$N_1(\theta)\propto\sin\theta$$
$$N_2(\phi)\propto\sin\phi,$$

where N_1 and N_2 are number densities as a function of angle and $\phi=\theta+\frac{1}{2}\pi$. Since $\sin\phi=\sin(\theta+\frac{1}{2}\pi)=\cos\theta$, we have

$$N_1(\theta)+N_2(\phi)\propto\sin\theta+\cos\theta.$$

This sum is approximately constant, which implies that if both coils are wound on the same form the number density around the circumference will also be approximately constant. The purpose of this is to obtain identical characteristics for both the X and Y coils.

Since magnetic fields obey the superposition principle, a field of any arbitrary orientation may be obtained by setting the respective currents in the X, Y, and Z coils proportional to the direction cosine of the desired orientation.

ACKNOWLEDGMENT

Conversations with Professor Elmar Zeitler are gratefully acknowledged.

*Work supported by a U.S. Public Health Service, NIH Grant #GM 19079.

[1]E. Mascart and J. Joubert, *Leçons sur l'Electricitê et le Magnêtisme*, Vol. I (Masson, Paris, 1882, 1896), p. 546.
[2]H.W. Watson and S.H. Burbury, *The Mathematical Theory of Electricity and Magnetism*, Vol. II (Oxford U.P., London, 1889), Secs. 318 and 449.
[3]L.A. Tseitlin, Sov. Phys.-Tech. Phys. **2**, 2592 (1957).
[4]L.A. Tseitlin, Sov. Phys.-Tech. Phys. **12**, 336 (1967).
[5]L.J. Laslett, J. Appl. Phys. **37**, 2361 (1966).
[6]J.A. Stratton, *Electromagnetic Theory* (McGraw-Hill, New York, 1941), Sec. 4.15.
[7]H.O.W. Richardson, Proc. Phys. Soc. (Lond.) B **65**, 5 (1952).
[8]E. Mascart and J. Joubert, in Ref. 1, p. 546.
[9]J.W.M. Du Mond, L. Bogart, J.L. Kohl, D.E. Muller, and J.R. Wilts, Special Technical Report No. 16, California Institute of Technology, Pasadena, Calif., 1952 (unpublished).
[10]T. Soller, M.A. Starr, and G.E. Valley, Jr., *Cathode Ray Tube Displays, Vol.* 22 *of Radiation Lab. Series* (McGraw-Hill, New York, 1948), Chap. 8 and Appendix B.
[11]D.M. Ginsberg and M.J. Melchner, Rev. Sci. Instrum. **41**, 122 (1970).
[12]R.K. Cacak, Rev. Sci. Instrum. **40**, 1468 (1969).