Self-Dual Gauge-Field Equations from a Differential Form Point of View

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Abstract

The utility of differential forms for understanding the origin of the self-dual gauge-field equations is illustrated by deriving the systems of linear partial differential equations introduced by Belavin and Zakharov and used in a different form by Ueno and Nakamura. The integrability condition for these systems of equations is then used to show their relation to a generalized form of the Ernst equation.

Key words: self-dual gauge-field equations, differential forms

1. INTRODUCTION

Beginning with the suggestive form of the Dirac equation for massless two-component fermions

$$(D_4 + i\,\mathbf{D}\cdot\boldsymbol{\sigma})u = 0, \tag{1a}$$

$$(D_4 - i\,\mathbf{D}\cdot\boldsymbol{\sigma})v = 0, \tag{1b}$$

Belavin and Zakharov[1] derived a system of linear partial differential equations, the "compatibility condition" for which is equivalent to the nonlinear self-duality relation for the Yang-Mills fields. Here, $D_i = \partial_i + A_i$, and the A_i are matrices obtained from the gauge potentials and generators of the associated Lie algebra. They noted that these equations correspond to the self-duality and anti-self-duality relations $F_{\mu\nu} = \pm {}^*F_{\mu\nu}$, respectively, and then looked for solutions of the form

$$v = \begin{pmatrix}\lambda\\ 1\end{pmatrix}\Psi(\lambda, x), \tag{2}$$

where $\Psi(\lambda, x)$ is a 2×2 matrix function belonging to the "isotopic space."

Substitution of Eq. (2) into Eq. (1b) yields the pair of linear partial differential equations

$$\begin{aligned} L_1\Psi &= [\lambda(D_2 - i\,D_1) + (D_4 + i\,D_3)]\Psi = 0 \\ L_2\Psi &= [\lambda(D_4 - i\,D_3) - (D_2 + i\,D_1)]\Psi = 0. \end{aligned} \tag{3}$$

As can be readily verified, the integrability condition $[L_1, L_2]\Psi = 0$ yields the anti-self-duality relation.

Ueno and Nakamura,[2] on the other hand, use the system

$$D_j\, Y(\zeta) = \zeta^{-1}A_j\, Y(\zeta),\ j = 1,\ 2, \tag{4}$$

where here

$$D_1 = \partial_y + \zeta^{-1}\partial_{\bar z},\quad A_1 = (\partial_{\bar z}J)J^{-1},$$

$$D_2 = -\partial_z + \zeta^{-1}\partial_{\bar y},\quad A_2 = (\partial_{\bar y}J)J^{-1},$$

and $Y(\zeta)$ is an $n \times n$ matrix function of $(y, \bar y, z, \bar z, \zeta)$, ζ being a complex "spectral" parameter. Function $J(y, \bar y, z, \bar z) = Y(0)$ is a $GL(n, C)$ matrix function satisfying

$$\partial_{\bar y}(J^{-1}\partial_y J) + \partial_{\bar z}(J^{-1}\partial_z J) = 0. \tag{5}$$

The latter equation represents the self-duality relation in a form introduced by Brihaye *et al.*[3]

The principal purpose of this paper is to illustrate how the use of differential forms can clarify the origin of the anti-self-dual gauge-field equations. It is also explicitly shown that the form of these equations used by Belavin and Zakharov and Ueno and Nakamura correspond to different representations of the self-duality relations. While this work is essentially didactical in nature, the use of differential forms allows the systems of linear partial differential equations used by these authors to be derived in a unified way.

The close connection, at least from a formal mathematical point of view, between static, axially symmetric gauge fields and the stationary, axially symmetric Einstein field equations is demonstrated in Sec. 5 by reducing the self-duality relation [Eq. (5)] to a generalized form of the Ernst equation.[4] The usual form of the Ernst equation then results from restricting consideration to the group $SU(2)$.

Before entering into the principal subject matter of this paper, some general comments on differential forms and complex spaces may prove useful. Let $\mathbf{C}^n$ denote the complex vector space of all n tuples $z = (z_1, \ldots, z_n)$ of complex numbers where $z_j = x_j + iy_j$ and x_j, $y_j \in \mathbf{R}$. The mapping $\alpha: \mathbf{C}^n \to \mathbf{R}^{2n}$ defined by $\alpha(z_1, \ldots, z_n) = (x_1, y_1, \ldots, x_n, y_n)$ can be used to identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$. In general, every n-dimensional complex manifold can be thought of as an orientable real analytic $2n$-dimensional manifold.

Let $n = 4$ so that the complex space of interest is $\mathbf{C}^4$. Also, let x_1, x_2, x_3, x_4 be complex orthonormal coordinates in $\mathbf{C}^4$, such that $x_i^2 = 1$ and $x_i \cdot x_j = 0$ for $i \neq j$. Then the coordinates $y, \bar{y}, z, \bar{z}$ introduced in Sec. 2 are null coordinates. Consider, for example, $y = (1/\sqrt{2})\ (x_1 + ix_2)$:

$$y^2 = \frac{1}{2}(x_1^2 + 2i\, x_1 \cdot x_2 - x_2^2) = 0, \tag{6}$$

since $x_1 \cdot x_2 = 0$ and $x_1^2 = x_2^2 = 1$.

In terms of the null coordinates $y,\ \bar{y},\ z,\ \bar{z}$ the exterior derivative of ω is written as

$$d\omega = \partial_y \omega dy + \partial_{\bar{y}} \omega d\bar{y} + \partial_z \omega dz + \partial_{\bar{z}} \omega d\bar{z}. \tag{7}$$

If ω is a p-form, $d\omega$ is a $(p+1)$-form. While the correspondence between $\mathbf{C}^n$ and $\mathbf{R}^{2n}$ can be used to introduce two additional operators ∂ and $\bar{\partial}$ such that $d = \partial + \bar{\partial}$, no use will be made of this decomposition in the following.

There is a close connection between $\mathbf{C}^4$ and the complexification **CM** of Minkowski space **M**. **CM** is the real eight-dimensional space obtained by allowing the coordinates t, x, y, z of **M** to be complex numbers. If **M** has signature -2, the metric in **CM** is the holomorphic or complex-analytic extension (rather than the Hermitian extension) of the Minkowski space metric; that is, it is still given by $dt^2 - dx^2 - dy^2 - dz^2$. The complex space $\mathbf{C}^4$ is isometric with that subspace of **CM** given by restricting t to real values and x, y, z to pure imaginary values. Null cones and lines are therefore well defined in $\mathbf{C}^4$ since **M** and $\mathbf{R}^4$ are contained in $\mathbf{C}^4$. Null vectors and hypersurfaces are introduced as generalizations of their real counterparts.

Throughout the following sections use is made of various forms of the duality or Hodge star operator *. While this operator is usually defined for real differential forms, the concept is readily extended to complex differential forms. The * operator maps p forms onto $(n-p)$-forms and has this property: the subspace of $^*\omega$ is orthogonal to the subspace of ω. Although the * operator is defined locally, it is independent of local coordinates but does depend on the existence of an inner product and on the orientation of the space.

Let n be the dimension of the space, λ a p-form and μ an $(n-p)$-form. Then for a suitably defined inner product, there is a unique $(n-p)$-form $^*\lambda$ such that

$$\lambda \wedge \mu = (^*\lambda \cdot \mu)\sigma, \tag{8}$$

where σ is the n-dimensional volume element of the space.

The duality operator can also be defined in terms of step products with the volume elements σ. (The Illinois Institute of Technology conventions are used for the step products; see Ref. 5.) The step product $\alpha \lceil \beta$ between a p-form α and a q-form β, where $q > p$, is defined as that $(q-p)$-form ω such that

$$\omega \cdot (\alpha \lceil \beta) = (\omega \wedge \alpha) \cdot \beta. \tag{9}$$

Replacing the q-form β with the n-form volume element σ and using the general relation $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$ results in

$$(\alpha \wedge \omega) \cdot \sigma = [(-1)^{p(n-p)} \alpha \lceil \sigma] \cdot \omega. \tag{10}$$

Comparing this result with the definition of the duality operator and restricting consideration to $\mathbf{C}^4$ leads to the identification

$$^*\alpha = (-1)^p \alpha \lceil \sigma. \tag{11}$$

The duality operator has important geometrical consequences in $\mathbf{C}^4$, where $^{**}\alpha = \alpha$. If the * operator is applied to 2-forms, the two eigenspaces corresponding to the eigenvalues $+1$ and -1 of the * operator give rise to the concept of self-duality and anti-self-duality.

2. A BASIS FOR SELF-DUAL AND ANTI-SELF-DUAL TWO-FORMS

In the following we will be concerned with 2-forms and would like to have a basis for such forms in terms of the null coordinates $y, \bar{y}, z, \bar{z}$. The latter can be introduced as follows: let x_i be complex rectangular coordinates in $\mathbf{C}^4$. The metric can then be written as

$$ds^2 = \sigma E \sigma^t, \tag{12}$$

where $\sigma = (x_1, x_2, x_3, x_4)$ and $E = \mathrm{diag}(1, 1, 1, 1)$. Under a unitary transformation $\sigma = \sigma' U$,

$$\begin{aligned} ds^2 &= \sigma' U E U^t \sigma'^t = \sigma' G \sigma'^t \\ G &:= U E U^t. \end{aligned} \tag{13}$$

If U has the form

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \end{pmatrix}, \tag{14}$$

G will be given by

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{15}$$

and since U is unitary,

$$\sigma' = \sigma U^\dagger = \frac{1}{\sqrt{2}}\,(x_1 + ix_2,\ x_1 - ix_2,\ x_3 + ix_4,\ x_3 - ix_4)\,. \quad (16)$$

The null coordinates $y,\ \bar{y},\ z,\ \bar{z}$ are then defined by

$$\begin{aligned} y &:= \frac{1}{\sqrt{2}}\,(x_1 + ix_2); \quad & z &:= \frac{1}{\sqrt{2}}\,(x_3 + ix_4); \\ \bar{y} &:= \frac{1}{\sqrt{2}}\,(x_1 - ix_2); \quad & \bar{z} &:= \frac{1}{\sqrt{2}}\,(x_3 - ix_4). \end{aligned} \quad (17)$$

Note that $\bar{y}$ is not the complex conjugate of y since the x_i are complex.

These coordinates arise naturally when considering the 2×2 matrix representation of a point in $\mathbf{C}^4$ given by

$$X := \frac{1}{\sqrt{2}}\, x^\mu \sigma_\mu = \frac{1}{\sqrt{2}}\,(i\, x_4 \sigma_0 + \mathbf{x} \cdot \sigma), \quad (18)$$

with σ_0 the 2×2 identity matrix and σ_i the Pauli spin matrices in the representation

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Writing out the expression for X, we obtain in terms of Eqs. (17)

$$X = \frac{1}{\sqrt{2}} \begin{pmatrix} x_3 + ix_4 & x_1 - ix_2 \\ x_1 + ix_2 & -(x_3 - ix_4) \end{pmatrix} = \begin{pmatrix} z & \bar{y} \\ y & -\bar{z} \end{pmatrix}. \quad (19)$$

Let $X^\dagger = \bar{X}^t$ (this becomes the usual Hermitian conjugate in real space):

$$X^\dagger = \begin{pmatrix} \bar{z} & \bar{y} \\ y & -z \end{pmatrix}; \quad (20)$$

$X \longrightarrow X^\dagger$ then corresponds to $x_4 \longrightarrow -x_4$.

The matrices X and $X^\dagger$ can be used to define the 1-form matrices dX and $dX^\dagger$. The 1-form elements of these matrices are such that the only nonzero step products[5] are

$$dy \lceil d\bar{y} = dz \lceil d\bar{z} = 1. \quad (21)$$

The exterior products between dX and $dX^\dagger$ (here and in the following the exterior or wedge product symbol $\wedge$ is omitted) can be used to define a basis for self-dual and anti-self-dual 2-forms:

$$dXdX^\dagger = \begin{pmatrix} dzd\bar{z} - dyd\bar{y} & 2\, dzd\bar{y} \\ 2\, dyd\bar{z} & dyd\bar{y} - dzd\bar{z} \end{pmatrix} \quad (22)$$

$$dX^\dagger dX = \begin{pmatrix} -(dzd\bar{z} + dyd\bar{y}) & 2\, d\bar{z}d\bar{y} \\ 2\, dydz & dyd\bar{y} + dzd\bar{z} \end{pmatrix}. \quad (23)$$

Here, the duality operation is defined as $^*\omega = -\omega \lceil dyd\bar{y}\ dzd\bar{z}$, where ω is one of the above two-form matrix elements, and $dx_1 dx_2 dx_3 dx_4 = -dyd\bar{y}dzd\bar{z}$. It can be seen that the elements of $dXdX^\dagger$ form a basis for anti-self-dual two-forms while the elements of $dX^\dagger dX$ form a basis for self-dual 2-forms.[6] Noting that the diagonal elements in both cases differ by an overall sign, a basis for self-dual and anti-self-dual 2-forms is given by

$$\begin{array}{cc} dydz & dyd\bar{z} \\ d\bar{y}d\bar{z} & d\bar{y}dz \\ dyd\bar{y} + dzd\bar{z} & dyd\bar{y} - dzd\bar{z} \\ \text{(self-dual)} & \text{(anti-self-dual)} \end{array} \quad (24)$$

3. THE GAUGE POTENTIAL AS A CONNECTION FORM

A gauge potential may be interpreted as a connection with an associated covariant derivative operator. Such an interpretation has been discussed by many authors and will therefore not be treated in detail here.[7]

Consider a four-dimensional complex, analytic and locally Euclidean Riemannian space M. At each point of M attach an n-dimensional complex vector space V^n. Let $\underline{\Psi} = (\underline{\Psi}_1,\ \underline{\Psi}_2,\ \ldots,\ \underline{\Psi}_n)$ be a basis for V^n and

$$\Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \vdots \\ \Psi^n \end{pmatrix}$$

be a basis for the dual space V^{n*}. In a new basis $\underline{\Psi}'$ is related to $\underline{\Psi}$ by the transformation $\underline{\Psi}' = \underline{\Psi}g$, where for the sake of generality it is assumed that[1] $g \,\varepsilon\, GL(n,\ C)$.

For a given gauge potential $A^a_\mu(x)$, define the connection form A by

$$A = A^a_\mu(x)\ \lambda_a dx^\mu, \quad (25)$$

where the λ_a are the generators of the gauge group. Then for a gauge potential to be interpreted as a connection form means that

$$d\underline{\Psi} = \underline{\Psi}A \quad (26)$$

or, alternatively, $d\Psi = -A\Psi$. The transformation property of the connection form A can be readily found from $\underline{\Psi}' = \underline{\Psi}g$,

$$\begin{aligned} d\underline{\Psi}' &= (d\underline{\Psi})g + \underline{\Psi}dg = \underline{\Psi}Ag + \underline{\Psi}dg = \underline{\Psi}(dg + Ag) \\ &= \underline{\Psi}'g^{-1}(dg + Ag) = \Psi'A', \end{aligned} \quad (27)$$

where

$$A' = g^{-1}dg + g^{-1}Ag \quad (28)$$

constitutes a gauge transformation.

The curvature 2-form F is readily obtained from Eq. (26)

$$\begin{aligned} d^2\underline{\Psi} &= (d\underline{\Psi})A + \underline{\Psi}dA = \underline{\Psi}AA + \underline{\Psi}dA = \underline{\Psi}(dA + AA) \\ F &:= dA + AA. \end{aligned} \quad (29)$$

4. THE SELF-DUALITY EQUATIONS

Ward,[8] drawing on twistor theory,[9] introduced the use of anti-self-dual planes in $\mathbf{C}^4$ to generate solutions to the Yang-Mills field equations. If $\underline{u}$ and $\underline{v}$ are two orthogonal tangent vectors to a plane in $\mathbf{C}^4$, and if the bivector $\underline{u} \wedge \underline{v}$ is anti-self-dual, then it defines a complex two-plane termed a β plane in twistor theory, while if it is self-dual it defines an α plane. Transformations between α and β planes can be effected by application of an element of $O(4, C)$ with determinant[10] equal to -1.

Ward pointed out the reason for defining β planes as follows: a self-dual gauge potential A_μ, may be interpreted as a connection with a covariant derivative operator $D_\mu = \partial_\mu + A_\mu$. Parallel propagation of a vector Ψ in the direction v^μ is then given by

$$v^\mu D_\mu \Psi = 0. \tag{30}$$

If propagation is constrained to lie in a β plane having v^μ as a tangent vector, this propagation law is integrable (the vector Ψ will return to its original value after propagation around a closed path).[11]

To make use of Ward's observations, note that in general the curvature 2-form F, defined by Eq. (29), can be decomposed into its self-dual and anti-self-dual parts $F = F^+ + F^-$, where

$$F^+ = \frac{1}{2}(F + {}^*F); \quad F^- = \frac{1}{2}(F - {}^*F).$$

Therefore, F will be self-dual ($F = {}^*F$) if and only if it vanishes when restricted to all β planes and anti-self-dual ($F = -{}^*F$) if and only if it vanishes when restricted to all α planes. Note that if F vanishes when restricted to all α planes and all β planes, then $F = 0$. The proof of this is given in Ref. 10 and proceeds along the following general lines: the tangent vectors to both α and β planes are always null vectors. Therefore, each pair of α and β planes through an arbitrary point $p \in \mathbf{C}^4$ intersect in a null line through p. The set of null lines determined by all pairs of α and β planes through p form a null cone through p. Now, if F vanishes when restricted to all α and β planes, then it vanishes along any null line contained in the null cone at p. Since one can always choose four linearly independent null vectors as a basis for $\mathbf{C}^4$, F vanishes at p. Hence, since p is an arbitrary point, F vanishes everywhere in $\mathbf{C}^4$.

The equations used by Belavin and Zakharov [Eqs. (3) or (2) and (1b)] and Ueno and Nakamura [Eqs. (4)] are derived by requiring that F vanish on the 2-form equivalent of Ward's α planes. This will make F anti-self-dual and is achieved by simply setting the self-dual parts of F, with respect to the basis of Eq. (24), equal to zero. To facilitate doing this, introduce the two-dimensional duality operation

$$\odot\omega = \omega \lceil (dydz + d\bar{y}d\bar{z}), \quad \odot\odot = -1, \tag{31}$$

and the projection operators π and $\bar{\pi}$, the latter defined by

$$\pi\omega = \pi(\omega_y dy + \omega_{\bar{y}} d\bar{y} + \omega_z dz + \omega_{\bar{z}} d\bar{z}) = \omega_y dy + \omega_z dz$$
$$\bar{\pi}\omega = \omega_{\bar{y}} d\bar{y} + \omega_{\bar{z}} d\bar{z}.$$

These operators obey the relations

$$\pi^2 = \pi, \ \bar{\pi}^2 = \bar{\pi}, \ \bar{\pi} + \pi = 1, \ \bar{\pi}\pi = 0$$
$$\pi\odot = \odot\bar{\pi}, \ \bar{\pi}\odot = \odot\pi.$$

The last two relations can be verified by operating on the 1-form ω; note that in general π, $\bar{\pi}$, and $\odot$ operate on 1-forms.

When applied to the exterior derivative d, the operations $\odot$, π, and $\bar{\pi}$ combine to yield

$$\pi d = dy\,\partial_y + dz\,\partial_z, \quad \odot\pi d = -d\bar{z}\,\partial_y + d\bar{y}\,\partial_z,$$
$$\bar{\pi} d = d\bar{y}\,\partial_{\bar{y}} + d\bar{z}\,\partial_{\bar{z}}, \quad \odot\bar{\pi} d = -dz\,\partial_{\bar{y}} + dy\,\partial_{\bar{z}},$$

since

$$\odot dy = -d\bar{z}; \quad \odot d\bar{y} = -dz;$$
$$\odot dz = d\bar{y}; \quad \odot d\bar{z} = dy.$$

Note that πd keeps $\bar{y}$ and $\bar{z}$ fixed while $\bar{\pi} d$ keeps y and z fixed.

In finding the self-dual components of $F = dA + AA$ it is important to remember that since the exterior product symbol $\wedge$ is omitted between forms, dA means $d \wedge A$. Anticipating that in a gauge where either $\bar{\pi}A = A$ and $\pi A = 0$, or $\pi A = A$ and $\bar{\pi}A = 0$, only some of the resulting equations are independent, we will not write out all the self-dual components of F, but rather illustrate the procedure, since the algebra is straightforward.

Consider the element $dydz$ of the self-dual basis given in Eq. (24). Two of the four possible equations that result from setting the self-dual components of F equal to zero are

$$\begin{aligned} (\pi d)(\pi A) + (\pi A)(\pi A) &= 0 \\ (\odot\bar{\pi} d)(\pi A) + (\odot\bar{\pi} A)(\pi A) &= 0. \end{aligned} \tag{32a}$$

Similarly, for the $d\bar{y}d\bar{z}$ element of the self-dual basis of Eq. (24) we choose

$$(\bar{\pi} d)(\bar{\pi} A) + (\bar{\pi} A)(\bar{\pi} A) = 0 \tag{33a}$$

$$(\odot\pi d)(\bar{\pi} A) + (\odot\pi A)(\bar{\pi} A) = 0. \tag{33b}$$

In the representations $\pi A = A$, $\bar{\pi} A = 0$ and $\bar{\pi} A = A$, $\pi A = 0$ these become, respectively,

$$\begin{array}{cc} \underline{\pi A = A, \ \bar{\pi} A = 0} & \underline{\bar{\pi} A = A, \ \pi A = 0} \\ (\pi d)A + AA = 0 & \bar{\pi} dA + AA = 0 \\ (\odot\bar{\pi} d)A = 0 & (\odot\pi d)A = 0 \end{array} \tag{34}$$

These constitute the residuum of independent relations in the chosen representation or gauge.

To show how these two representations arise, we will need to use the theorem of Frobenius[12]:

Theorem: If A is a 1-form matrix and $dA = A^2$, then there exists a matrix of functions M such that $A = (dM)M^{-1}$.

Equations (32a) and (33a) therefore imply that

$$\pi A = -(\pi dM)M^{-1}; \quad \bar{\pi} A = -(\bar{\pi} dN)N^{-1}. \tag{35}$$

Multiplying on the left by $\underline{\Psi}$ and on the right by M and N, respectively, these become

$$\pi\underline{\Psi}AM + \underline{\Psi}\pi dM = 0; \quad \bar{\pi}\underline{\Psi}AN + \underline{\Psi}\bar{\pi} dN = 0. \tag{36}$$

Noting that $d\underline{\Psi} = \underline{\Psi}A$, the latter pair of equations can be written as

$$\pi d(\underline{\Psi}M) = 0; \quad \bar{\pi} d(\underline{\Psi}N) = 0. \tag{37}$$

Define the matrix $J := M^{-1}N$. The two representations (choice of gauge for A) are then arrived at as follows in Sec. 4.1 and Sec. 4.2.

4.1 The Representation $\bar{\pi}A = A$, $\pi A = 0$

In general,

$$d(\underline{\Psi}M) = \bar{\pi} d(\underline{\Psi}M) + \pi d(\underline{\Psi}M). \tag{38}$$

The second term on the RHS is zero by the first of Eqs. (37), while the first term is

$$\bar{\pi} d(\underline{\Psi}M) = \bar{\pi} d(\underline{\Psi}NJ^{-1}) = \bar{\pi} d(\underline{\Psi}N)J^{-1} + \underline{\Psi}N\bar{\pi} dJ^{-1}. \tag{39}$$

Now, the first term on the RHS of Eq. (39) vanishes by the second of Eqs. (37) so that

$$d(\underline{\Psi}M) = \underline{\Psi}N\bar{\pi} dJ^{-1} = \underline{\Psi}MJ\bar{\pi} dJ^{-1} = -\underline{\Psi}M\bar{\pi} dJ \cdot J^{-1}. \tag{40}$$

Letting $\underline{\Psi}' = \underline{\Psi}M$ and comparing with $d\underline{\Psi}' = \underline{\Psi}'A'$ we have

$$A' = -\bar{\pi} dJ \cdot J^{-1}. \tag{41}$$

Dropping the prime, in this choice of gauge $\bar{\pi}A = A$, $\pi A = 0$.

4.2 The Representation $\pi A = A$, $\bar{\pi}A = 0$

Here, we begin with

$$d(\underline{\Psi}N) = \bar{\pi} d(\underline{\Psi}N) + \pi d(\underline{\Psi}N). \tag{42}$$

The first term on the RHS vanishes by the second of Eq. (37), while the second term is

$$\pi d(\underline{\Psi}N) = \pi d(\underline{\Psi}MJ) = \pi d(\underline{\Psi}M)J + \underline{\Psi}M\pi dJ. \tag{43}$$

The first term on the RHS vanishes by the first of Eq. (37) and therefore,

$$d(\underline{\Psi}N) = \underline{\Psi}M\pi dJ = \underline{\Psi}NJ^{-1}\pi dJ = -\underline{\Psi}N\pi dJ^{-1} \cdot J. \tag{44}$$

Again, letting $\underline{\Psi}' = \underline{\Psi}N$ and comparing with $d\underline{\Psi}' = \underline{\Psi}'A'$,

$$A' = -\pi dJ^{-1} \cdot J. \tag{45}$$

Dropping the prime, in this choice of gauge $\pi A = A$, $\bar{\pi}A = 0$.

4.3. The Equations of Ueno and Nakamura

In the representation $\bar{\pi}A = A$, $\pi A = 0$ the requirement that the self-dual parts of F vanish led to the second pair of Eqs. (34). These equations are arbitrary up to multiplication by a complex number ζ. Combining the pair gives

$$(\bar{\pi} d - \zeta \odot \pi d)A + AA = 0 \tag{46}$$

or

$$DA + AA = 0, \tag{47}$$

where

$$D := \bar{\pi} d - \zeta \odot \pi d. \tag{48}$$

If we now define $\Gamma := -A = \bar{\pi} dJ \cdot J^{-1}$, Eq. (47) becomes $D\Gamma = \Gamma\Gamma$. Then by the Frobenius integration theorem there exists an invertible matrix function Y such that $\Gamma = (DY)Y^{-1}$ or

$$DY = \Gamma Y. \tag{49}$$

To see that this corresponds to the equations of Ueno and Nakamura, use the explicit form of D given by

$$D = d\bar{y}(\partial_{\bar{y}} - \zeta\partial_z) + d\bar{z}(\partial_{\bar{z}} + \zeta\partial_y) \tag{50}$$

and $\Gamma = \bar{\pi} dJ \cdot J^{-1}$. Substitution and multiplication by ζ^{-1} yields

$$d\bar{y}[(-\partial_z + \zeta^{-1}\partial_{\bar{y}}) - \zeta^{-1}\partial_{\bar{y}}J \cdot J^{-1}]Y$$

$$+ d\bar{z}[(\partial_y + \zeta^{-1}\partial_{\bar{z}}) - \zeta^{-1}\partial_{\bar{z}}J \cdot J^{-1}]Y = 0. \tag{51}$$

Defining

$$D_1 = \partial_y + \zeta^{-1}\partial_{\bar{z}}, \quad A_1 = \partial_{\bar{z}}J \cdot J^{-1},$$

$$D_2 = -\partial_z + \zeta^{-1}\partial_{\bar{y}}, \quad A_2 = \partial_{\bar{y}}J \cdot J^{-1},$$

Eq. (51) becomes

$$d\bar{y}[D_2Y - \zeta^{-1}A_2Y] + d\bar{z}[D_1Y - \zeta^{-1}A_1Y] = 0, \tag{52}$$

from which we get the pair of Eqs. (4). Note that

$$\partial_z A_1 + \partial_y A_2 = \partial_z(\partial_{\bar{z}}J \cdot J^{-1}) + \partial_y(\partial_{\bar{y}}J \cdot J^{-1}) = 0, \tag{53}$$

since $\pi A = 0$, and Eq. (53) is equivalent to Eq. (5).

4.4 The Equations of Belavin and Zakharov

Here, we combine the first pair of Eqs. (34) to obtain

$$(\pi d - \zeta \odot \bar{\pi} d)A + AA = 0 \tag{54}$$

or $DA + AA = 0$ where $D := \pi d - \zeta \odot \bar{\pi} d$ and no distinction is made between the D defined here and in Eq. (48); no confusion should result.

Again, defining $\Gamma := -A = \pi dJ^{-1} \cdot J$ we have $D\Gamma = \Gamma\Gamma$ and by the Frobenius integration theorem there exists an invertible matrix function Y such that $\Gamma = (DY)Y^{-1}$ or $DY = \Gamma Y$. In this case the explicit representation of D is given by

$$D = dy(\partial_y - \zeta\partial_{\bar{z}}) + dz(\partial_z + \zeta\partial_{\bar{y}}). \tag{55}$$

Substitution of Eq. (55) and $\Gamma = \pi dJ^{-1} \cdot J$ into $DY = \Gamma Y$ gives

$$dy[(-\partial_{\bar{z}} + \zeta^{-1}\partial y) - \zeta^{-1}\partial_y J^{-1} \cdot J]Y$$

$$+ dz[(\partial_{\bar{y}} + \zeta^{-1}\partial_z) - \zeta^{-1}\partial_z J^{-1} \cdot J]Y = 0. \tag{56}$$

For consistency with Ling-Lie Chau *et al.*[13] we define

$$D_1 = \partial_{\bar{y}} + \zeta^{-1}\partial_z; \quad A_1 = \partial_z J^{-1} \cdot J = -J^{-1}\partial_z J;$$
$$D_2 = \partial_{\bar{z}} - \zeta^{-1}\partial_y; \quad A_2 = -\partial_y J^{-1} \cdot J = J^{-1}\partial_y J. \tag{57}$$

And Eq. (56) is equivalent to Eq. (4). Here $\bar{\pi}A = 0$ so that $\partial_{\bar{z}}A_1 - \partial_{\bar{y}}A_2 = 0$ again yields Eq. (5).

To see that Eqs. (4) and (56) are indeed equivalent to the form given by Belavin and Zakharov in Eqs. (1b) or (3), note that if $\pi A = A$ satisfies Eq. (54) then so does $(\pi - \zeta \odot \bar{\pi})A = A$. The components of A are then

$$A = (\pi - \zeta \odot \bar{\pi})(A_y dy + A_z dz + A_{\bar{y}} d\bar{y} + A_{\bar{z}} d\bar{z})$$
$$= (A_y - \zeta A_{\bar{z}})dy + (A_z + \zeta A_{\bar{y}})dz. \tag{58}$$

Now, from Eq. (45) we have

$$A = -\pi dJ^{-1} \cdot J = -(dy\partial_y + dz\partial_z)J^{-1} \cdot J$$
$$= -dy\partial_y J^{-1} \cdot J - dz\partial_z J^{-1} \cdot J = dyA_2 - dzA_1. \tag{59}$$

Comparing Eqs. (58) and (59),

$$A_1 = -(A_z + \zeta A_{\bar{y}}); \quad A_2 = (A_y - \zeta A_{\bar{z}}). \tag{60}$$

Eqs. (4) and (56) then become

$$(\partial_{\bar{y}} + \zeta^{-1}\partial_z)Y = -\zeta^{-1}(A_z + \zeta A_{\bar{y}})Y$$
$$(\partial_{\bar{z}} - \zeta^{-1}\partial_y)Y = \zeta^{-1}(A_y - \zeta A_{\bar{z}})Y. \tag{61}$$

Letting $\lambda = \zeta^{-1}$ and $Y = \Psi$ this pair of equations takes the form

$$(\partial_{\bar{y}} + A_{\bar{y}})\Psi + \lambda(\partial_z + A_z)\Psi = 0$$
$$(\partial_{\bar{z}} + A_{\bar{z}})\Psi - \lambda(\partial_y + A_y)\Psi = 0. \tag{62}$$

Noting the definitions given in Eqs. (17), $\partial_y = \partial_{x_1} + i\,\partial_{x_2}$, $A_y = A_{x_1} + i\,A_{x_2}$, etc., Eqs. (62) can be written as

$$(D_1 - iD_2)\Psi + \lambda(D_3 + iD_4)\Psi = 0$$
$$(D_3 - iD_4)\Psi - \lambda(D_1 + iD_2)\Psi = 0, \tag{63}$$

where here, as in Eqs. (3), $D_i = \partial_{x_i} + A_{x_i}$. These are the same as Eqs. (3).

The equations of Ueno and Nakamura and Belavin and Zakharov are therefore seen to arise naturally in the language of exterior forms. They correspond to the equations obtained by setting the self-dual parts of the curvature form $F = dA + AA$ equal to zero in the different gauges characterized by $\bar{\pi}A = A$, $\pi A = 0$ and $\pi A = A$, $\bar{\pi}A = 0$.

The derivation given above is an example of the utility of the language of modern differential geometry (differential forms, fiber bundles, etc.) for studying gauge fields. While space-time has the global symmetries that correspond to the Poincaré group, gauge theories have additional internal symmetries that give extra degrees of freedom. These are captured in the Lie group G identified with the "gauge group" and its associated Lie algebra, g. It is the interpretation of the gauge potential as a connection form with an associated covariant derivative operator that leads to the strong geometrical content of gauge theory.

In the case of the gauge fields described by the equations of Belavin and Zakharov and Ueno and Nakamura, the gauge potential [Eq. (25)] is a g-valued 2-form on $\mathbf{C}^4$, two gauge potentials being regarded as equivalent if they are related by a gauge transformation [Eq. (28)]. The corresponding gauge field or curvatures [(Eq. (29)] is a g-valued 1-form on $\mathbf{C}^4$. The requirement that this curvature 2-form vanish on the 2-form equivalent of the α planes described in Sec. 4 led directly to both forms of self-duality relations introduced by these authors.

5. A GENERALIZED FORM OF THE ERNST EQUATION

For the important case of axial symmetry, the Einstein field equations of general relativity can be reformulated in terms of a complex function E, independent of azimuth. Ernst[4] obtained this result by beginning with the Papapetrou line element in the form

$$ds^2 = f^{-1}[e^{2\gamma}(dz^2 + d\rho^2) + \rho^2 d\phi^2] - f(dt - \omega d\phi)^2, \tag{64}$$

where f, ω, and γ are only functions of z and ρ (the symbols here have their conventional meanings).

The field equations for f and ω are then obtained by variation from the Lagrangian density

$$L = -\frac{1}{2}\rho f^{-2}\nabla f \cdot \nabla f + \frac{1}{2}\rho^{-1}f^2\nabla\omega \cdot \nabla\omega. \tag{65}$$

This results in

$$f\nabla^2 f = \nabla f \cdot \nabla f - \rho^{-2}f^4\nabla\omega \cdot \nabla\omega$$
$$\nabla \cdot (\rho^{-2}f^2\nabla\omega) = 0, \tag{66}$$

where ∇ is the three-dimensional divergence operator. Introducing a function ξ independent of azimuth, Ernst was able to show that the second of Eqs. (66) can be satisfied identically. Expressing the first of Eqs. (66) in terms of ξ and introducing the complex function $E = f + i\xi$ leads to the homogeneous quadratic differential equation

$$(\mathrm{Re}\,E)\nabla^2 E = \nabla E \cdot \nabla E. \tag{67}$$

It is interesting, and highly suggestive, that there is a close connection between the self-duality relation as expressed by Eq. (5) and the Ernst equation, Eq. (67). To illustrate this relationship, a generalized form of the Ernst equation will first be derived from the self-duality relation Eq. (5) by imposing axial symmetry and requiring that the fields be static. The generalized equation will then be reduced to the usual form of the Ernst equation by using the Poincaré parametrization for J.

Equation (5) can be written in terms of the complex rectangular coordinates x_i as

$$J^{-1}\Box J + \nabla J^{-1} \cdot \nabla J + i[(\partial_{x_4}J^{-1}\partial_{x_3}J - \partial_{x_3}J^{-1}\partial_{x_4}J)$$
$$+ (\partial_{x_2}J^{-1}\partial_{x_1}J - \partial_{x_1}J^{-1}\partial_{x_2}J)] = 0, \tag{68}$$

where $\Box$ and ∇ are, respectively, the four-dimensional Laplacian and gradient operator for Euclidean space.

Equation (68) is more general than that given by Takeno,[14] who identifies terms similar to the bracketed last term of Eq. (68), but restricts consideration to $SU(2)$ in Yang's[15] R gauge.

If the static condition $\partial_{x_4} J = 0$ holds, the first term within the square brackets of Eq. (68) will vanish, while if axial symmetry with respect to the x_3-axis is imposed, the last term within the brackets will vanish. Thus the static, axially symmetric case results in

$$J^{-1}\nabla^2 J + \nabla J^{-1}\cdot\nabla J = 0, \tag{69}$$

where ∇^2 and ∇ are the three-dimensional Laplacian and gradient operators. Equation (69) can be regarded as a generalized Ernst equation.[16]

Restricting consideration to $SU(2)$ gauge fields, Pohlmeyer[17] has noted that if the Poincaré parametrization for J is chosen, following Yang, to be of the form

$$J = \frac{1}{\Psi}\begin{pmatrix} 1 & \bar{\rho} \\ \rho & \Psi^2 + \rho\bar{\rho} \end{pmatrix}, \tag{70}$$

the self-duality relation, Eq. (5), results in Yang's equations in the R gauge.[18] The latter have been shown by Witten[19] to reduce to the Ernst equation in the static, axially symmetric case for constant phase solutions and real x_i where $\bar{\rho} = \rho^*$ and Ψ is real (ρ^* is the complex conjugate of ρ).

Similarly, using this parametrization for J in Eq. (69) results in the two equations

$$\Psi\nabla^2\rho - 2\nabla\rho\cdot\nabla\Psi = 0$$

$$\Psi\nabla^2\Psi + \nabla\rho\cdot\nabla\rho^* - \nabla\Psi\cdot\nabla\Psi - (1+\Psi^{-2}\rho\rho^*)^{-1}[\rho\nabla^2\rho^* - \rho^*\nabla^2\rho$$
$$+ 2\Psi^{-1}(\rho^*\nabla\rho\cdot\nabla\Psi - \rho\nabla\rho^*\cdot\nabla\Psi)] = 0. \tag{71}$$

For constant phase solutions (where $\rho = \sigma e^{i\alpha}$, σ being a real function and α a real constant), these become

$$\begin{aligned} \Psi\nabla^2\sigma - 2\nabla\sigma\cdot\nabla\Psi &= 0 \\ \Psi\nabla^2\Psi + \nabla\sigma\cdot\nabla\sigma - \nabla\Psi\cdot\nabla\Psi &= 0. \end{aligned} \tag{72}$$

Defining $E = \Psi + i\sigma$ these combine to yield the form of the Ernst equation given by

$$(\mathrm{Re}\, E)\nabla^2 E - \nabla E\cdot\nabla E = 0. \tag{73}$$

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Résumé

L'utilité des formes différentielles pour comprendre l'origine des équations du champ de jauge auto-duale est illustrée en dérivant le système d'équations différentielles linéaires aux dérivées partielles introduit par Belavin et utilisé dans une forme différente par Ueno et Nakamura. La condition d'integrabilité pour ces systèmes est alors utilisé pour montrer leur relation avec une forme généralisée de l'équation d'Ernst.

Endnote

[1] Restrictions on g change the group; for example, if det $g = 1$, $g \in SL(n, C)$; introduction of an inner product such that $\langle \underline{\Psi}', \underline{\Psi}' \rangle = \langle \underline{\Psi}, \underline{\Psi} \rangle$ implies that $gg^{\dagger} = 1$ (g is unitary), while a restriction to orthonormal bases $\langle \Psi^{\alpha}, \underline{\Psi}_{\beta} \rangle = \delta^{\alpha}_{\beta}$ means that A is anti-Hermitian.

References

1. A.A. Belavin and V.E. Zakharov, Phys. Lett. B **73**, 53 (1978).
2. K. Ueno and Y. Nakamura, Phys. Lett. B **109**, 273 (1982).
3. Y. Brihaye, D.B. Fairlie, J. Nuyts, and R.G. Yates, J. Math. Phys. **19**, 2528 (1978).
4. F.J. Ernst, Phys. Rev. **167**, 1175 (1968).
5. For a discussion of Grassmann inner products, see I. Hauser and F.J. Ernst, J. Math. Phys. **19**, 1316 (1978), and F.J. Ernst and J. Plebanski, Ann. Phys. **107**, 266 (1977). Of particular use here is the relation

$$a \lceil (b \wedge c) = b(a \lceil c) + (-1)^r (a \lceil b) c,$$

where a is a 1-form and b, c are p and r-forms, respectively. Note, also, that $(a \wedge b) \lceil c = a \lceil (b \lceil c)$, for a, b, c any p, q, r forms.
6. See Ref. 10 for a discussion from a somewhat different point of view.
7. See, for example, Ref. 10 and W. Drechsler and M.E. Mayer, *Fiber Bundle Techniques in Gauge Theories, Lecture Notes in Physics* (Springer-Verlag, Berlin, 1977), Vol. 67.
8. R.S. Ward, Phys. Lett. A **61**, 81 (1977).
9. R. Penrose and R.S. Ward, *General Relativity and Gravitation*, edited by A. Held (Plenum Press, NY, 1980), vol. 2. Chap. 7; see, also, R. Penrose and M.A.H. MacCallum, Phys. Rep. C **6**, 241 (1972).
10. C. Nash and S. Sen, *Topology and Geometry for Physicists* (Academic Press, Orlando, Fla. 1983), p. 283.
11. In this context see the discussion given by Yang.[15]
12. H. Flanders, *Differential Forms with Applications to the Physical Science* (Academic Press, NY, 1963), Secs. 7.3 and 7.4.
13. Ling-Lie Chau, Ge Mo-Lin, and Wu Yong-Shi, Phys. Rev. D **25**, 1086 (1982).
14. S. Takeno, Prog. Theor. Phys. **66**, 1250 (1981).
15. C.N. Yang, Phys. Rev. Lett. **38**, 1377 (1977).
16. In the context of SU(N) and Yang's R-gauge, this equation has been given by B.C. Xanthopoulos, *Solutions of Einstein's Equations: Techniques and Results, Lecture Notes in Physics* (Springer-Verlag, Berlin, 1984), Vol. 205.
17. K. Pohlmeyer, Commun. Math. Phys. **72**, 37 (1980).
18. The parametrization given in Eq. (70) has been discussed in the context of SL(2, C) and Minkowski space fields by H.J. de Vega, Commun. Math. Phys. **116**, 659 (1988).
19. L. Witten, Phys. Rev. D **19**, 718 (1979).

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